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Wave Equations on Lorentzian Manifolds and Quantization

European Mathematical Society

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Preface

In General Relativity spacetime is described mathematically by a Lorentzian manifold. Gravitation manifests itself as the curvature of this manifold. Physical fields, such as the electromagnetic field, are defined on this manifold and have to satisfy a wave equation. This book provides an introduction to the theory of linear wave equations on Lorentzian manifolds. In contrast to other texts on this topic, [Friedlander1975], [Günther1988], we develop the global theory. This means, we ask for existence and uniqueness of solutions which are defined on all of the underlying manifold. Such results are of great importance and are already used much in the literature despite the fact that published proofs are hard to find. Tracing back the references one typically ends at Leray's unpublished lecture notes [Leray1953] or their exposition [Choquet-Bruhat1968].

In this text we develop the global theory from scratch in a modern geometric language. In the first chapter we provide basic definitions and facts about distributions on manifolds, Lorentzian geometry, and normally hyperbolic operators. We study the building blocks for local solutions, the Riesz distributions, in some detail. In the second chapter we show how to solve wave equations locally. Using Riesz distributions and a formal recursive procedure one first constructs formal fundamental solutions. These are formal series solving the equations formally, but in general they do not converge. Using suitable cut-offs one gets "almost solutions" from these formal solutions. They are well-defined distributions but solve the equation only up to an error term. This is then corrected by some further analysis which yields true local fundamental solutions.

This procedure is similar to the construction of the heat kernel for a Laplace type operator on a compact Riemannian manifold. The analogy goes even further. Similar to the short-time asymptotics for the heat kernel, the formal fundamental solution turns out to be an asymptotic expansion of the true fundamental solution. Along the diagonal the coefficients of this asymptotic expansion are given by the same algebraic expression in the curvature of the manifold, the coefficients of the operator, and their derivatives as the heat kernel coefficients.

In the third chapter we use the local theory to study global solutions. This means we construct global fundamental solutions, Green's operators, and solutions to the Cauchy problem. This requires assumptions on the geometry of the underlying manifold. In Lorentzian geometry one has to deal with the problem that there is no good analog for the notion of completeness of Riemannian manifolds. In our context globally hyperbolic manifolds turn out to be the right class of manifolds to consider. Most basic models in General Relativity turn out to be globally hyperbolic but there are exceptions such as anti-deSitter spacetime. This is why we also include a section in which we study cases where one can guarantee existence (but not uniqueness) of global solutions on certain non-globally hyperbolic manifolds.

In the last chapter we apply the analytical results and describe the basic mathematical concepts behind field quantization. The aim of quantum field theory on curved spacetimes is to provide a partial unification of General Relativity with Quantum Physics where the gravitational field is left classical while the other fields are quantized. We develop the theory of C^*-algebras and CCR-representations in full detail to the extent that we need. Then we construct the quantization functors and check that the Haag–Kastler axioms of a local quantum field theory are satisfied. We also construct the Fock space and the quantum field.

From a physical perspective we just enter the door to quantum field theory but do not go very far. We do not discuss n-point functions, states, renormalization, nonlinear fields, nor physical applications such as Hawking radiation. For such topics we refer to the corresponding literature. However, this book should provide the reader with a firm mathematical basis to enter this fascinating branch of physics.

In the appendix we collect background material on category theory, functional analysis, differential geometry, and differential operators that is used throughout the text. This collection of material is included for the convenience of the reader but cannot replace a thorough introduction to these topics. The reader should have some experience with differential geometry. Despite the fact that normally hyperbolic operators on Lorentzian manifolds look formally exactly like Laplace type operators on Riemannian manifolds their analysis is completely different. The elliptic theory of Laplace type operators is not needed anywhere in this text. All results on hyperbolic equations which are relevant to the subject are developed in full detail. Therefore no prior knowledge of the theory of partial differential equations is needed.

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Potsdam, February 2007 Christian Bär, Nicolas Ginoux, and Frank Pfäffle

Contents

1 Preliminaries

We want to study solutions to wave equations on Lorentzian manifolds. In this first chapter we develop the basic concepts needed for this task. In the appendix the reader will find the background material on differential geometry, functional analysis and other fields of mathematics that will be used throughout this text without further comment.

A wave equation is given by a certain differential operator of second order called a "normally hyperbolic operator". In general, these operators act on sections in vector bundles which is the geometric way of saying that we are dealing with systems of equations and not just with scalar equations. It is important to allow that the sections may have certain singularities. This is why we work with distributional sections rather than with smooth or continuous sections only.

The concept of distributions on manifolds is explained in the first section. One nice feature of distributions is the fact that one can apply differential operators to them and again obtain a distribution without any further regularity assumption.

The simplest example of a normally hyperbolic operator on a Lorentzian manifold is given by the d'Alembert operator on Minkowski space. Its fundamental solution, a concept to be explained later, can be described explicitly. This gives rise to a family of distributions on Minkowski space, the Riesz distributions, which will provide the building blocks for solutions in the general case later.

After explaining the relevant notions from Lorentzian geometry we will show how to "transplant" Riesz distributions from the tangent space into the Lorentzian manifold. We will also derive the most important properties of the Riesz distributions.

1.1 Distributions on manifolds

Let us start by giving some definitions and by fixing the terminology for distributions on manifolds. We will confine ourselves to those facts that we will actually need later on. A systematic and much more complete introduction may be found e.g. in [Friedlander1998].

1.1.1 Preliminaries on distributions. Let M be a manifold equipped with a smooth volume density dV. Later on we will use the volume density induced by a Lorentzian metric but this is irrelevant for now. We consider a real or complex vector bundle $E \to M$. We will always write $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether E is real or complex. The space of compactly supported smooth sections in E will be denoted by $\mathcal{D}(M, E)$. We equip E and T^*M with connections, both denoted by ∇. They induce connections on the tensor bundles $T^*M \otimes \cdots \otimes T^*M \otimes E$, again denoted by ∇. For a continuously differentiable section $\varphi \in C^1(M, E)$ the covariant derivative is a continuous section in $T^*M \otimes E$, $\nabla\varphi \in C^0(M, T^*M \otimes E)$. More generally, for $\varphi \in C^k(M, E)$ we get $\nabla^k\varphi \in C^0(M, \underbrace{T^*M \otimes \cdots \otimes T^*M}_{k \text{ factors}} \otimes E)$.

We choose a Riemannian metric on T^*M and a Riemannian or Hermitian metric on E depending on whether E is real or complex. This induces metrics on all bundles $T^*M \otimes \cdots \otimes T^*M \otimes E$. Hence the norm of $\nabla^k \varphi$ is defined at all points of M.

For a subset $A \subset M$ and $\varphi \in C^k(M, E)$ we define the C^k*-norm* by

$$\|\varphi\|_{C^k(A)} := \max_{j=0,\ldots,k} \sup_{x \in A} |\nabla^j \varphi(x)|. \tag{1.1}$$

If A is compact, then different choices of metric and connection yield equivalent norms $\|\cdot\|_{C^k(A)}$. For this reason there will usually be no need to explicitly specify the metrics and the connections.

The elements of $\mathcal{D}(M, E)$ are referred to as *test sections* in E. We define a notion of convergence of test sections.

Definition 1.1.1. Let $\varphi, \varphi_n \in \mathcal{D}(M, E)$. We say that the sequence $(\varphi_n)_n$ *converges to* φ *in* $\mathcal{D}(M, E)$ if the following two conditions hold:

(1) There is a compact set $K \subset M$ such that the supports of all φ_n are contained in K, i.e., $\mathrm{supp}(\varphi_n) \subset K$ for all n.

(2) The sequence $(\varphi_n)_n$ converges to φ in all C^k-norms over K, i.e., for each $k \in \mathbb{N}$

$$\|\varphi - \varphi_n\|_{C^k(K)} \xrightarrow[n\to\infty]{} 0.$$

We fix a finite-dimensional $\mathbb{K}$-vector space W. Recall that $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether E is real or complex.

Definition 1.1.2. A $\mathbb{K}$-linear map $F : \mathcal{D}(M, E^*) \to W$ is called a *distribution in* E *with values in* W if it is continuous in the sense that for all convergent sequences $\varphi_n \to \varphi$ in $\mathcal{D}(M, E^*)$ one has $F[\varphi_n] \to F[\varphi]$. We write $\mathcal{D}'(M, E, W)$ for the space of all W-valued distributions in E.

Note that since W is finite-dimensional all norms $|\cdot|$ on W yield the same topology on W. Hence there is no need to specify a norm on W for Definition 1.1.2 to make sense. Note moreover, that distributions in E act on test sections in E^*.

Lemma 1.1.3. *Let* F *be a* W*-valued distribution in* E *and let* $K \subset M$ *be compact. Then there is a nonnegative integer* k *and a constant* $C > 0$ *such that for all* $\varphi \in \mathcal{D}(M, E^*)$ *with* $\mathrm{supp}(\varphi) \subset K$ *we have*

$$|F[\varphi]| \leq C \cdot \|\varphi\|_{C^k(K)}. \tag{1.2}$$

The smallest k for which inequality (1.2) holds is called the *order* of F over K.

Proof. Assume (1.2) does not hold for any pair of C and k. Then for every positive integer k we can find a nontrivial section $\varphi_k \in \mathcal{D}(M, E^*)$ with $\mathrm{supp}(\varphi_k) \subset K$ and $|F[\varphi_k]| \geq k \cdot \|\varphi_k\|_{C^k}$. We define sections $\psi_k := \frac{1}{|F[\varphi_k]|}\varphi_k$. Obviously, these ψ_k satisfy $\mathrm{supp}(\psi_k) \subset K$ and

$$\|\psi_k\|_{C^k(K)} = \frac{1}{|F[\varphi_k]|}\|\varphi_k\|_{C^k(K)} \leq \frac{1}{k}.$$

Hence for $k \geq j$

$$\|\psi_k\|_{C^j(K)} \leq \|\psi_k\|_{C^k(K)} \leq \frac{1}{k}.$$

Therefore the sequence $(\psi_k)_k$ converges to 0 in $\mathcal{D}(M, E^*)$. Since F is a distribution we get $F[\psi_k] \to F[0] = 0$ for $k \to \infty$. On the other hand, $|F[\psi_k]| = \left|\frac{1}{|F[\varphi_k]|} F[\varphi_k]\right| = 1$ for all k, which yields a contradiction. □

Lemma 1.1.3 states that the restriction of any distribution to a (relatively) compact set is of finite order. We say that a distribution F is of order m if m is the smallest integer such that for each compact subset $K \subset M$ there exists a constant C so that

$$|F[\varphi]| \leq C \cdot \|\varphi\|_{C^m(K)}$$

for all $\varphi \in \mathcal{D}(M, E^*)$ with $\mathrm{supp}(\varphi) \subset K$. Such a distribution extends uniquely to a continuous linear map on $\mathcal{D}^m(M, E^*)$, the space of C^m-sections in E^* with compact support. Convergence in $\mathcal{D}^m(M, E^*)$ is defined similarly to that of test sections. We say that φ_n converge to φ in $\mathcal{D}^m(M, E^*)$ if the supports of the φ_n and φ are contained in a common compact subset $K \subset M$ and $\|\varphi - \varphi_n\|_{C^m(K)} \to 0$ as $n \to \infty$.

Next we give two important examples of distributions.

Example 1.1.4. Pick a bundle $E \to M$ and a point $x \in M$. The *delta-distribution* δ_x is an E_x^*-valued distribution in E. For $\varphi \in \mathcal{D}(M, E^*)$ it is defined by

$$\delta_x[\varphi] = \varphi(x).$$

Clearly, δ_x is a distribution of order 0.

Example 1.1.5. Every locally integrable section $f \in L^1_{\mathrm{loc}}(M, E)$ can be interpreted as a $\mathbb{K}$-valued distribution in E by setting for any $\varphi \in \mathcal{D}(M, E^*)$

$$f[\varphi] := \int_M \varphi(f)\, \mathrm{dV}.$$

As a distribution f is of order 0.

Lemma 1.1.6. *Let M and N be differentiable manifolds equipped with smooth volume densities. Let $E \to M$ and $F \to N$ be vector bundles. Let $K \subset N$ be compact and let $\varphi \in C^k(M \times N, E \boxtimes F^*)$ be such that* $\mathrm{supp}(\varphi) \subset M \times K$. *Let $m \leq k$ and let $T \in \mathcal{D}'(N, F, \mathbb{K})$ be a distribution of order m. Then the map*

$$\begin{aligned} f : M &\to E, \\ x &\mapsto T[\varphi(x, \cdot)], \end{aligned}$$

defines a C^{k-m}-section in E with support contained in the projection of $\mathrm{supp}(\varphi)$ *to the first factor, i.e.,* $\mathrm{supp}(f) \subset \{x \in M \mid$ *there exists* $y \in K$ *such that* $(x, y) \in \mathrm{supp}(\varphi)\}$. *In particular, if φ is smooth with compact support, and T is any distribution in F, then f is a smooth section in E with compact support.*

Moreover, x-derivatives up to order $k-m$ may be interchanged with T. More precisely, if P is a linear differential operator of order $\le k-m$ acting on sections in E, then

$$Pf = T[P_x\varphi(x,\cdot)].$$

Here $E \boxtimes F^*$ denotes the vector bundle over $M \times N$ whose fiber over $(x,y) \in M \times N$ is given by $E_x \otimes F_y^*$.

Proof. There is a canonical isomorphism

$$E_x \otimes \mathcal{D}^k(N, F^*) \to \mathcal{D}^k(N, E_x \otimes F^*),$$
$$v \otimes s \mapsto (y \mapsto v \otimes s(y)).$$

Thus we can apply $\mathrm{id}_{E_x} \otimes T$ to $\varphi(x,\cdot) \in \mathcal{D}^k(N, E_x \otimes F^*) \cong E_x \otimes \mathcal{D}^k(N, F^*)$ and we obtain $(\mathrm{id}_{E_x} \otimes T)[\varphi(x,\cdot)] \in E_x$. We briefly write $T[\varphi(x,\cdot)]$ instead of $(\mathrm{id}_{E_x} \otimes T)[\varphi(x,\cdot)]$.

To see that the section $x \mapsto T[\varphi(x,\cdot)]$ in E is of regularity C^{k-m} we may assume that M is an open ball in $\mathbb{R}^p$ and that the vector bundle $E \to M$ is trivialized over M, $E = M \times \mathbb{K}^n$, because differentiability and continuity are local properties.

For fixed $y \in N$ the map $x \mapsto \varphi(x,y)$ is a C^k-map $U \to \mathbb{K}^n \otimes F_y^*$. We perform a Taylor expansion at $x_0 \in U$, see [Friedlander1998, p. 38f]. For $x \in U$ we get

$$\begin{aligned}
\varphi(x,y) &= \sum_{|\alpha|\le k-m-1} \tfrac{1}{\alpha!} D_x^\alpha \varphi(x_0,y)(x-x_0)^\alpha \\
&\quad + \sum_{|\alpha|=k-m} \frac{k-m}{\alpha!} \int_0^1 (1-t)^{k-m-1} D_x^\alpha \varphi((1-t)x_0+tx,y)(x-x_0)^\alpha \, dt \\
&= \sum_{|\alpha|\le k-m} \tfrac{1}{\alpha!} D_x^\alpha \varphi(x_0,y)(x-x_0)^\alpha \\
&\quad + \sum_{|\alpha|=k-m} \tfrac{k-m}{\alpha!} \int_0^1 (1-t)^{k-m-1} \big(D_x^\alpha \varphi((1-t)x_0+tx,y) \\
&\qquad\qquad - D_x^\alpha \varphi(x_0,y)\,\big)\, dt \cdot (x-x_0)^\alpha.
\end{aligned}$$

Here we used the usual multi-index notation, $\alpha = (\alpha_1,\dots,\alpha_p) \in \mathbb{N}^p$, $|\alpha| = \alpha_1+\dots+\alpha_p$, $D_x^\alpha = \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}\dots(\partial x^p)^{\alpha_p}}$, and $x^\alpha = x_1^{\alpha_1}\dots x_p^{\alpha_p}$. For $|\alpha| \le k-m$ we certainly have $D_x^\alpha\varphi(\cdot,\cdot) \in C^m(U \times N, \mathbb{K}^n \otimes F^*)$ and, in particular, $D_x^\alpha\varphi(x_0,\cdot) \in \mathcal{D}^m(N, \mathbb{K}^n \otimes F^*)$. We apply T to get

$$\begin{aligned}
T[\varphi(x,\cdot)] &= \sum_{|\alpha|\le k-m} \tfrac{1}{\alpha!} T[D_x^\alpha\varphi(x_0,\cdot)](x-x_0)^\alpha \\
&\quad + \sum_{|\alpha|=k-m} \tfrac{k-m}{\alpha!} T\Big[\int_0^1 (1-t)^{k-m-1} \big(D_x^\alpha\varphi((1-t)x_0+tx,\cdot) \\
&\qquad\qquad - D_x^\alpha\,\varphi(x_0,\cdot)\big)\, dt\Big](x-x_0)^\alpha.
\end{aligned} \tag{1.3}$$

Restricting the x to a compact convex neighborhood $U' \subset U$ of x_0 the $D_x^\alpha \varphi(\cdot,\cdot)$ and all their y-derivatives up to order m are *uniformly* continuous on $U' \times K$. Given $\varepsilon > 0$ there exists $\delta > 0$ so that $|\nabla_y^j D_x^\alpha \varphi(\tilde{x}, y) - \nabla_y^j D_x^\alpha \varphi(x_0, y)| \leq \frac{\varepsilon}{m+1}$ whenever $|\tilde{x} - x_0| \leq \delta$, $j = 0, \dots, m$. Thus for x with $|x - x_0| \leq \delta$

$$\begin{aligned}
&\left\| \int_0^1 (1-t)^{k-m-1} \left(D_x^\alpha \varphi((1-t)x_0 + tx, \cdot) - D_x^\alpha \varphi(x_0, \cdot) \right) dt \right\|_{C^m(M)} \\
&\quad = \left\| \int_0^1 (1-t)^{k-m-1} \left(D_x^\alpha \varphi((1-t)x_0 + tx, \cdot) - D_x^\alpha \varphi(x_0, \cdot) \right) dt \right\|_{C^m(K)} \\
&\quad \leq \int_0^1 (1-t)^{k-m-1} \left\| D_x^\alpha \varphi((1-t)x_0 + tx, \cdot) - D_x^\alpha \varphi(x_0, \cdot) \right\|_{C^m(K)} dt \\
&\quad \leq \int_0^1 (1-t)^{k-m-1} \varepsilon \, dt \\
&\quad = \frac{\varepsilon}{k-m}.
\end{aligned}$$

Since T is of order m this implies in (1.3) that $T[\int_0^1 \dots dt] \to 0$ as $x \to x_0$. Therefore the map $x \mapsto T[\varphi(x,\cdot)]$ is $k - m$ times differentiable with derivatives $D_x^\alpha|_{x=x_0} T[\varphi(x,\cdot)] = T[D_x^\alpha \varphi(x_0,\cdot)]$. The same argument also shows that these derivatives are continuous in x. □

1.1.2 Differential operators acting on distributions. Let E and F be two $\mathbb{K}$-vector bundles over the manifold M, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Consider a linear differential operator $P : C^\infty(M, E) \to C^\infty(M, F)$. There is a unique linear differential operator $P^* : C^\infty(M, F^*) \to C^\infty(M, E^*)$ called the *formal adjoint* of P such that for any $\varphi \in \mathcal{D}(M, E)$ and $\psi \in \mathcal{D}(M, F^*)$

$$\int_M \psi(P\varphi) \, \mathrm{dV} = \int_M (P^*\psi)(\varphi) \, \mathrm{dV}. \tag{1.4}$$

If P is of order k, then so is P^* and (1.4) holds for all $\varphi \in C^k(M, E)$ and $\psi \in C^k(M, F^*)$ such that $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi)$ is compact. With respect to the canonical identification $E = (E^*)^*$ we have $(P^*)^* = P$.

Any linear differential operator $P : C^\infty(M, E) \to C^\infty(M, F)$ extends canonically to a linear operator $P : \mathcal{D}'(M, E, W) \to \mathcal{D}'(M, F, W)$ by

$$(PT)[\varphi] := T[P^*\varphi]$$

where $\varphi \in \mathcal{D}(M, F^*)$. If a sequence $(\varphi_n)_n$ converges in $\mathcal{D}(M, F^*)$ to 0, then the sequence $(P^*\varphi_n)_n$ converges to 0 as well because P^* is a differential operator. Hence $(PT)[\varphi_n] = T[P^*\varphi_n] \to 0$. Therefore PT is again a distribution.

The map $P : \mathcal{D}'(M, E, W) \to \mathcal{D}'(M, F, W)$ is $\mathbb{K}$-linear. If P is of order k and φ is a C^k-section in E, seen as a $\mathbb{K}$-valued distribution in E, then the distribution $P\varphi$ coincides with the continuous section obtained by applying P to φ classically.

The case when P is of order 0, i.e., $P \in C^\infty(M, \mathrm{Hom}(E, F))$, is of special importance. Then $P^* \in C^\infty(M, \mathrm{Hom}(F^*, E^*))$ is the pointwise adjoint. In particular, for a function $f \in C^\infty(M)$ we have

$$(fT)[\varphi] = T[f\varphi].$$

1.1.3 Supports

Definition 1.1.7. The *support* of a distribution $T \in \mathcal{D}'(M, E, W)$ is defined as the set

$$\mathrm{supp}(T) := \{x \in M \mid \text{for all neighborhoods } U \text{ of } x \text{ there exists} \\ \varphi \in \mathcal{D}(M, E) \text{ with } \mathrm{supp}(\varphi) \subset U \text{ and } T[\varphi] \neq 0\}.$$

It follows from the definition that the support of T is a closed subset of M. In case T is a L^1_{loc}-section this notion of support coincides with the usual one for sections.

If for $\varphi \in \mathcal{D}(M, E^*)$ the supports of φ and T are disjoint, then $T[\varphi] = 0$. Namely, for each $x \in \mathrm{supp}(\varphi)$ there is a neighborhood U of x such that $T[\psi] = 0$ whenever $\mathrm{supp}(\psi) \subset U$. Cover the compact set $\mathrm{supp}(\varphi)$ by finitely many such open sets $U_1, \dots, U_k$. Using a partition of unity one can write $\varphi = \psi_1 + \cdots + \psi_k$ with $\psi_j \in \mathcal{D}(M, E^*)$ and $\mathrm{supp}(\psi_j) \subset U_j$. Hence

$$T[\varphi] = T[\psi_1 + \cdots + \psi_k] = T[\psi_1] + \cdots + T[\psi_k] = 0.$$

Be aware that it is not sufficient to assume that φ vanishes on $\mathrm{supp}(T)$ in order to ensure $T[\varphi] = 0$. For example, if $M = \mathbb{R}$ and E is the trivial $\mathbb{K}$-line bundle let $T \in \mathcal{D}'(\mathbb{R}, \mathbb{K})$ be given by $T[\varphi] = \varphi'(0)$. Then $\mathrm{supp}(T) = \{0\}$ but $T[\varphi] = \varphi'(0)$ may well be nonzero while $\varphi(0) = 0$.

If $T \in \mathcal{D}'(M, E, W)$ and $\varphi \in C^\infty(M, E^*)$, then the evaluation $T[\varphi]$ can be defined if $\mathrm{supp}(T) \cap \mathrm{supp}(\varphi)$ is compact even if the support of φ itself is noncompact. To do this pick a function $\sigma \in \mathcal{D}(M, \mathbb{R})$ that is constant 1 on a neighborhood of $\mathrm{supp}(T) \cap \mathrm{supp}(\varphi)$ and put

$$T[\varphi] := T[\sigma\varphi].$$

This definition is independent of the choice of σ since for another choice σ' we have

$$T[\sigma\varphi] - T[\sigma'\varphi] = T[(\sigma - \sigma')\varphi] = 0$$

because $\mathrm{supp}((\sigma - \sigma')\varphi)$ and $\mathrm{supp}(T)$ are disjoint.

Let $T \in \mathcal{D}'(M, E, W)$ and let $\Omega \subset M$ be an open subset. Each test section $\varphi \in \mathcal{D}(\Omega, E^*)$ can be extended by 0 and yields a test section $\varphi \in \mathcal{D}(M, E^*)$. This defines an embedding $\mathcal{D}(\Omega, E^*) \subset \mathcal{D}(M, E^*)$. By the restriction of T to Ω we mean its restriction from $\mathcal{D}(M, E^*)$ to $\mathcal{D}(\Omega, E^*)$.

Definition 1.1.8. The *singular support* $\mathrm{sing\,supp}(T)$ of a distribution $T \in \mathcal{D}'(M, E, W)$ is the set of points which do not have a neighborhood restricted to which T coincides with a smooth section.

The singular support is also closed and we always have $\mathrm{sing\,supp}(T) \subset \mathrm{supp}(T)$.

Example 1.1.9. For the delta-distribution δ_x we have $\mathrm{supp}(\delta_x) = \mathrm{sing\,supp}(\delta_x) = \{x\}$.

1.1.4 Convergence of distributions. The space $\mathcal{D}'(M, E)$ of distributions in E will always be given the *weak topology*. This means that $T_n \to T$ in $\mathcal{D}'(M, E, W)$ if and only if $T_n[\varphi] \to T[\varphi]$ for all $\varphi \in \mathcal{D}(M, E^*)$. Linear differential operators P are always continuous with respect to the weak topology. Namely, if $T_n \to T$, then we have for every $\varphi \in \mathcal{D}(M, E^*)$

$$PT_n[\varphi] = T_n[P^*\varphi] \to T[P^*\varphi] = PT[\varphi].$$

Hence

$$PT_n \to PT.$$

Lemma 1.1.10. *Let $T_n, T \in C^0(M, E)$ and suppose $\|T_n - T\|_{C^0(M)} \to 0$. Consider T_n and T as distributions.*

Then $T_n \to T$ in $\mathcal{D}'(M, E)$. In particular, for every linear differential operator P we have $PT_n \to PT$.

Proof. Let $\varphi \in \mathcal{D}(M, E)$. Since $\|T_n - T\|_{C^0(M)} \to 0$ and $\varphi \in L^1(M, E)$, it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
\lim_{n\to\infty} T_n[\varphi] &= \lim_{n\to\infty} \int_M T_n(x) \cdot \varphi(x)\, \mathrm{dV}(x) \\
&= \int_M \lim_{n\to\infty} (T_n(x) \cdot \varphi(x))\, \mathrm{dV}(x) \\
&= \int_M (\lim_{n\to\infty} T_n(x)) \cdot \varphi(x)\, \mathrm{dV}(x) \\
&= \int_M T(x) \cdot \varphi(x)\, \mathrm{dV}(x) \\
&= T[\varphi].
\end{aligned}$$

□

1.1.5 Two auxiliary lemmas. The following situation will arise frequently. Let E, F, and G be $\mathbb{K}$-vector bundles over M equipped with metrics and with connections which we all denote by ∇. We give $E \otimes F$ and $F^* \otimes G$ the induced metrics and connections. Here and henceforth F^* will denote the dual bundle to F. The natural pairing $F \otimes F^* \to \mathbb{K}$ given by evaluation of the second factor on the first yields a vector bundle homomorphism $E \otimes F \otimes F^* \otimes G \to E \otimes G$ which we write as $\varphi \otimes \psi \mapsto \varphi \cdot \psi$.[1]

Lemma 1.1.11. *For all C^k-sections φ in $E \otimes F$ and ψ in $F^* \otimes G$ and all $A \subset M$ we have*

$$\|\varphi \cdot \psi\|_{C^k(A)} \leq 2^k \cdot \|\varphi\|_{C^k(A)} \cdot \|\psi\|_{C^k(A)}.$$

Proof. The case $k = 0$ follows from the Cauchy–Schwarz inequality. Namely, for fixed $x \in M$ we choose an orthonormal basis f_i, $i = 1, \ldots, r$, for F_x. Let f_i^* be

[1] If one identifies $E \otimes F$ with $\mathrm{Hom}(E^*, F)$ and $F^* \otimes G$ with $\mathrm{Hom}(F, G)$, then $\varphi \cdot \psi$ corresponds to $\psi \circ \varphi$.

the basis of F_x^* dual to f_i. We write $\varphi(x) = \sum_{i=1}^r e_i \otimes f_i$ for suitable $e_i \in E_x$ and similarly $\psi(x) = \sum_{i=1}^r f_i^* \otimes g_i$, $g_i \in G_x$. Then $\varphi(x) \cdot \psi(x) = \sum_{i=1}^r e_i \otimes g_i$ and we see

$$\begin{aligned}
|\varphi(x) \cdot \psi(x)|^2 = \Big|\sum_{i=1}^r e_i \otimes g_i\Big|^2 &= \sum_{i,j=1}^r \langle e_i \otimes g_i, e_j \otimes g_j\rangle = \sum_{i,j=1}^r \langle e_i, e_j\rangle\langle g_i, g_j\rangle \\
&\leq \sqrt{\sum_{i,j=1}^r \langle e_i, e_j\rangle^2} \cdot \sqrt{\sum_{i,j=1}^r \langle g_i, g_j\rangle^2} \\
&\leq \sqrt{\sum_{i,j=1}^r |e_i|^2|e_j|^2} \cdot \sqrt{\sum_{i,j=1}^r |g_i|^2|g_j|^2} \\
&= \sqrt{\sum_{i=1}^r |e_i|^2 \sum_{j=1}^r |e_j|^2} \cdot \sqrt{\sum_{i=1}^r |g_i|^2 \sum_{j=1}^r |g_j|^2} \\
&= \sum_{i=1}^r |e_i|^2 \cdot \sum_{i=1}^r |g_i|^2 \\
&= |\varphi(x)|^2 \cdot |\psi(x)|^2.
\end{aligned}$$

Now we proceed by induction on k.

$$\begin{aligned}
\|\nabla^{k+1}(\varphi \cdot \psi)\|_{C^0(A)} &\leq \|\nabla(\varphi \cdot \psi)\|_{C^k(A)} \\
&= \|(\nabla\varphi) \cdot \psi + \varphi \cdot \nabla\psi\|_{C^k(A)} \\
&\leq \|(\nabla\varphi) \cdot \psi\|_{C^k(A)} + \|\varphi \cdot \nabla\psi\|_{C^k(A)} \\
&\leq 2^k \cdot \|\nabla\varphi\|_{C^k(A)} \cdot \|\psi\|_{C^k(A)} + 2^k \cdot \|\varphi\|_{C^k(A)} \cdot \|\nabla\psi\|_{C^k(A)} \\
&\leq 2^k \cdot \|\varphi\|_{C^{k+1}(A)} \cdot \|\psi\|_{C^{k+1}(A)} \\
&\qquad + 2^k \cdot \|\varphi\|_{C^{k+1}(A)} \cdot \|\psi\|_{C^{k+1}(A)} \\
&= 2^{k+1} \cdot \|\varphi\|_{C^{k+1}(A)} \cdot \|\psi\|_{C^{k+1}(A)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\varphi \cdot \psi\|_{C^{k+1}(A)} &= \max\{\|\varphi \cdot \psi\|_{C^k(A)}, \|\nabla^{k+1}(\varphi \cdot \psi)\|_{C^0(A)}\} \\
&\leq \max\{2^k \cdot \|\varphi\|_{C^k(A)} \cdot \|\psi\|_{C^k(A)}, 2^{k+1} \cdot \|\varphi\|_{C^{k+1}(A)} \cdot \|\psi\|_{C^{k+1}(A)}\} \\
&= 2^{k+1} \cdot \|\varphi\|_{C^{k+1}(A)} \cdot |\psi\|_{C^{k+1}(A)}.
\end{aligned}$$ ☐

This lemma allows us to estimate the C^k-norm of products of sections in terms of the C^k-norms of the factors. The next lemma allows us to deal with compositions of functions. We recursively define the following universal constants:

$$\alpha(k, 0) := 1, \quad \alpha(k, j) := 0$$

for $j > k$ and for $j < 0$, and

$$\alpha(k+1, j) := \max\{\alpha(k, j),\, 2^k \cdot \alpha(k, j-1)\} \tag{1.5}$$

if $1 \le j \le k$. The precise values of the $\alpha(k, j)$ are not important. The definition was made in such a way that the following lemma holds.

Lemma 1.1.12. *Let Γ be a real valued C^k-function on a Lorentzian manifold M and let $\sigma : \mathbb{R} \to \mathbb{R}$ be a C^k-function. Then for all $A \subset M$ and $I \subset \mathbb{R}$ such that $\Gamma(A) \subset I$ we have*

$$\|\sigma \circ \Gamma\|_{C^k(A)} \le \|\sigma\|_{C^k(I)} \cdot \max_{j=0,\dots,k} \alpha(k, j) \|\Gamma\|^j_{C^k(A)}.$$

Proof. We again perform an induction on k. The case $k = 0$ is obvious. By Lemma 1.1.11

$$\begin{aligned}
\|\nabla^{k+1}(\sigma \circ \Gamma)\|_{C^0(A)} &= \|\nabla^k[(\sigma' \circ \Gamma) \cdot \nabla\Gamma]\|_{C^0(A)} \\
&\le \|(\sigma' \circ \Gamma) \cdot \nabla\Gamma\|_{C^k(A)} \\
&\le 2^k \cdot \|\sigma' \circ \Gamma\|_{C^k(A)} \cdot \|\nabla\Gamma\|_{C^k(A)} \\
&\le 2^k \cdot \|\sigma' \circ \Gamma\|_{C^k(A)} \cdot \|\Gamma\|_{C^{k+1}(A)} \\
&\le 2^k \cdot \|\sigma'\|_{C^k(I)} \cdot \max_{j=0,\dots,k} \alpha(k, j) \|\Gamma\|^j_{C^{k+1}(A)} \cdot \|\Gamma\|_{C^{k+1}(A)} \\
&\le 2^k \cdot \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=0,\dots,k} \alpha(k, j) \|\Gamma\|^{j+1}_{C^{k+1}(A)} \\
&= 2^k \cdot \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=1,\dots,k+1} \alpha(k, j-1) \|\Gamma\|^j_{C^{k+1}(A)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\sigma \circ \Gamma\|_{C^{k+1}(A)} &= \max\{\|\sigma \circ \Gamma\|_{C^k(A)}, \|\nabla^{k+1}(\sigma \circ \Gamma)\|_{C^0(A)}\} \\
&\le \max\{\|\sigma\|_{C^k(I)} \cdot \max_{j=0,\dots,k} \alpha(k, j) \|\Gamma\|^j_{C^k(A)}, \\
&\qquad 2^k \cdot \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=1,\dots,k+1} \alpha(k, j-1) \|\Gamma\|^j_{C^{k+1}(A)}\} \\
&\le \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=0,\dots,k+1} \max\{\alpha(k, j), 2^k \alpha(k, j-1)\} \|\Gamma\|^j_{C^{k+1}(A)} \\
&= \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=0,\dots,k+1} \alpha(k+1, j) \|\Gamma\|^j_{C^{k+1}(A)}.
\end{aligned}$$

□

1.2 Riesz distributions on Minkowski space

The distributions $R_+(\alpha)$ and $R_-(\alpha)$ to be defined below were introduced by M. Riesz in the first half of the 20th century in order to find solutions to certain differential equations. He collected his results in [Riesz1949]. We will derive all relevant facts in full detail.

Let V be an n-dimensional real vector space, let $\langle\cdot,\cdot\rangle$ be a nondegenerate symmetric bilinear form of index 1 on V. Hence $(V, \langle\cdot,\cdot\rangle)$ is isometric to n-dimensional Minkowski space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_0)$ where $\langle x, y\rangle_0 = -x_1y_1 + x_2y_2 + \cdots + x_ny_n$. Set

$$\gamma\colon V \to \mathbb{R}, \quad \gamma(X) := -\langle X, X\rangle. \tag{1.6}$$

A nonzero vector $X \in V \setminus \{0\}$ is called *timelike* (or *lightlike* or *spacelike*) if and only if $\gamma(X) > 0$ (or $\gamma(X) = 0$ or $\gamma(X) < 0$ respectively). The zero vector $X = 0$ is considered as spacelike. The set $I(0)$ of timelike vectors consists of two connected components. We choose a *time-orientation* on V by picking one of these two connected components. Denote this component by $I_+(0)$ and call its elements *future directed*. Put $J_+(0) := \overline{I_+(0)}$, $C_+(0) := \partial I_+(0)$, $I_-(0) := -I_+(0)$, $J_-(0) := -J_+(0)$, and $C_-(0) := -C_+(0)$.

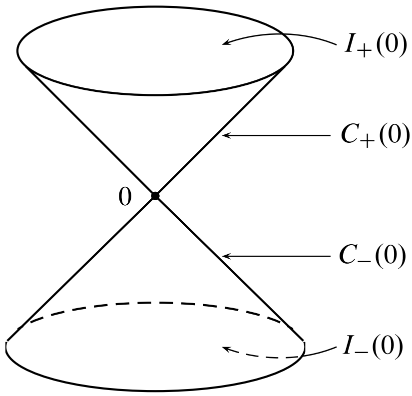


Figure 1. Light cone in Minkowski space.

Definition 1.2.1. For any complex number α with $\Re e(\alpha) > n$ let $R_+(\alpha)$ and $R_-(\alpha)$ be the complex-valued continuous functions on V defined by

$$R_\pm(\alpha)(X) := \begin{cases} C(\alpha, n)\gamma(X)^{\frac{\alpha-n}{2}}, & \text{if } X \in J_\pm(0), \\ 0, & \text{otherwise,} \end{cases}$$

where $C(\alpha, n) := \frac{2^{1-\alpha}\pi^{\frac{2-n}{2}}}{(\frac{\alpha}{2}-1)!(\frac{\alpha-n}{2})!}$ and $z \mapsto (z-1)!$ is the Gamma function.

For $\alpha \in \mathbb{C}$ with $\Re e(\alpha) \le n$ this definition no longer yields continuous functions due to the singularities along $C_\pm(0)$. This requires a more careful definition of $R_\pm(\alpha)$ as a distribution which we will give below. Even for $\Re e(\alpha) > n$ we will from now on consider the continuous functions $R_\pm(\alpha)$ as distributions as explained in Example 1.1.5.

Since the Gamma function has no zeros the map $\alpha \mapsto C(\alpha, n)$ is holomorphic on $\mathbb{C}$. Hence for each fixed testfunction $\varphi \in \mathcal{D}(V, \mathbb{C})$ the map $\alpha \mapsto R_\pm(\alpha)[\varphi]$ yields a holomorphic function on $\{\Re e(\alpha) > n\}$.

There is a natural differential operator $\Box$ acting on functions on V, $\Box f := \partial_{e_1}\partial_{e_1} f - \partial_{e_2}\partial_{e_2} f - \cdots - \partial_{e_n}\partial_{e_n} f$ where $e_1, \ldots, e_n$ is any basis of V such that $-\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \cdots = \langle e_n, e_n\rangle = 1$ and $\langle e_i, e_j\rangle = 0$ for $i \neq j$. Such a basis $e_1, \ldots, e_n$ is called *Lorentzian orthonormal*. The operator $\Box$ is called the *d'Alembert operator*. The formula in Minkowski space with respect to the standard basis may look more familiar to the reader,

$$\Box = \frac{\partial^2}{(\partial x^1)^2} - \frac{\partial^2}{(\partial x^2)^2} - \cdots - \frac{\partial^2}{(\partial x^n)^2}.$$

The definition of the d'Alembertian on general Lorentzian manifolds can be found in the next section. In the following lemma the application of differential operators such as $\Box$ to the $R_\pm(\alpha)$ is to be taken in the distributional sense.

Lemma 1.2.2. *For all $\alpha \in \mathbb{C}$ with $\Re e(\alpha) > n$ we have*

(1) $\gamma \cdot R_\pm(\alpha) = \alpha(\alpha - n + 2) R_\pm(\alpha + 2)$,
(2) $(\operatorname{grad} \gamma) \cdot R_\pm(\alpha) = 2\alpha \ \operatorname{grad}\ R_\pm(\alpha + 2)$,
(3) $\Box R_\pm(\alpha + 2) = R_\pm(\alpha)$.
(4) *The map $\alpha \mapsto R_\pm(\alpha)$ extends uniquely to $\mathbb{C}$ as a holomorphic family of distributions. In other words, for each $\alpha \in \mathbb{C}$ there exists a unique distribution $R_\pm(\alpha)$ on V such that for each testfunction φ the map $\alpha \mapsto R_\pm(\alpha)[\varphi]$ is holomorphic.*

Proof. Identity (1) follows from

$$\frac{C(\alpha, n)}{C(\alpha + 2, n)} = \frac{2^{(1-\alpha)}\ (\frac{\alpha+2}{2} - 1)!\ (\frac{\alpha+2-n}{2})!}{2^{(1-\alpha-2)}\ (\frac{\alpha}{2} - 1)!\ (\frac{\alpha-n}{2})!} = \alpha\,(\alpha - n + 2).$$

To show (2) we choose a Lorentzian orthonormal basis $e_1, \ldots, e_n$ of V and we denote differentiation in direction e_i by ∂_i. We fix a testfunction φ and integrate by parts:

$$\begin{aligned}
\partial_i \gamma \cdot R_\pm(\alpha)[\varphi] &= C(\alpha, n) \int_{J_\pm(0)} \gamma(X)^{\frac{\alpha-n}{2}}\, \partial_i \gamma(X) \varphi(X)\, dX \\
&= \frac{2C(\alpha, n)}{\alpha + 2 - n} \int_{J_\pm(0)} \partial_i (\gamma(X)^{\frac{\alpha-n+2}{2}}) \varphi(X)\, dX \\
&= -2\alpha C(\alpha + 2, n) \int_{J_\pm(0)} \gamma(X)^{\frac{\alpha-n+2}{2}}\, \partial_i \varphi(X)\, dX \\
&= -2\alpha R_\pm(\alpha + 2)[\partial_i \varphi] \\
&= 2\alpha \partial_i R_\pm(\alpha + 2)[\varphi],
\end{aligned}$$

which proves (2). Furthermore, it follows from (2) that

$$\begin{aligned}\partial_i^2 R_\pm(\alpha+2) &= \partial_i\left(\frac{1}{2\alpha}\partial_i\gamma\cdot R_\pm(\alpha)\right)\\ &= \frac{1}{2\alpha}\left(\partial_i^2\gamma\cdot R_\pm(\alpha)+\partial_i\gamma\cdot\left(\frac{1}{2(\alpha-2)}\partial_i\gamma\cdot R_\pm(\alpha-2)\right)\right)\\ &= \frac{1}{2\alpha}\partial_i^2\gamma\cdot R_\pm(\alpha)+\frac{1}{4\alpha(\alpha-2)}(\partial_i\gamma)^2\frac{(\alpha-2)(\alpha-n)}{\gamma}\cdot R_\pm(\alpha)\\ &= \left(\frac{1}{2\alpha}\partial_i^2\gamma+\frac{\alpha-n}{4\alpha}\cdot\frac{(\partial_i\gamma)^2}{\gamma}\right)\cdot R_\pm(\alpha),\end{aligned}$$

so that

$$\begin{aligned}\Box\, R_\pm(\alpha+2) &= \left(\frac{n}{\alpha}+\frac{\alpha-n}{4\alpha}\cdot\frac{4\gamma}{\gamma}\right)R_\pm(\alpha)\\ &= R_\pm(\alpha).\end{aligned}$$

To show (4) we first note that for fixed $\varphi\in\mathscr{D}(V,\mathbb{C})$ the map $\{\Re e(\alpha)>n\}\to\mathbb{C}$, $\alpha\mapsto R_\pm(\alpha)[\varphi]$, is holomorphic. For $\Re e(\alpha)>n-2$ we set

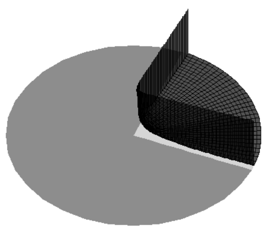
$$\widetilde{R}_\pm(\alpha) := \Box\, R_\pm(\alpha+2). \tag{1.7}$$

This defines a distribution on V. The map $\alpha\mapsto\widetilde{R}_\pm(\alpha)$ is then holomorphic on $\{\Re e(\alpha)>n-2\}$. By (3) we have $\widetilde{R}_\pm(\alpha)=R_\pm(\alpha)$ for $\Re e(\alpha)>n$, so that $\alpha\mapsto\widetilde{R}_\pm(\alpha)$ extends $\alpha\mapsto R_\pm(\alpha)$ holomorphically to $\{\Re e(\alpha)>n-2\}$. We proceed inductively and construct a holomorphic extension of $\alpha\mapsto R_\pm(\alpha)$ on $\{\Re e(\alpha)>n-2k\}$ (where $k\in\mathbb{N}\setminus\{0\}$) from that on $\{\Re e(\alpha)>n-2k+2\}$ just as above. Note that these extensions necessarily coincide on their common domain since they are holomorphic and they coincide on an open subset of $\mathbb{C}$. We therefore obtain a holomorphic extension of $\alpha\mapsto R_\pm(\alpha)$ to the whole of $\mathbb{C}$, which is necessarily unique. □

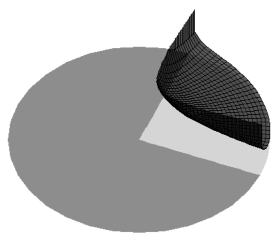
Lemma 1.2.2 (4) defines $R_\pm(\alpha)$ for all $\alpha\in\mathbb{C}$, not as functions but as distributions.

Definition 1.2.3. We call $R_+(\alpha)$ the *advanced Riesz distribution* and $R_-(\alpha)$ the *retarded Riesz distribution* on V for $\alpha\in\mathbb{C}$.

The following illustration shows the graphs of Riesz distributions $R_+(\alpha)$ for $n=2$ and various values of α. In particular, one sees the singularities along $C_+(0)$ for $\Re e(\alpha)\le 2$.



$\alpha = 0.1$



$\alpha = 1$

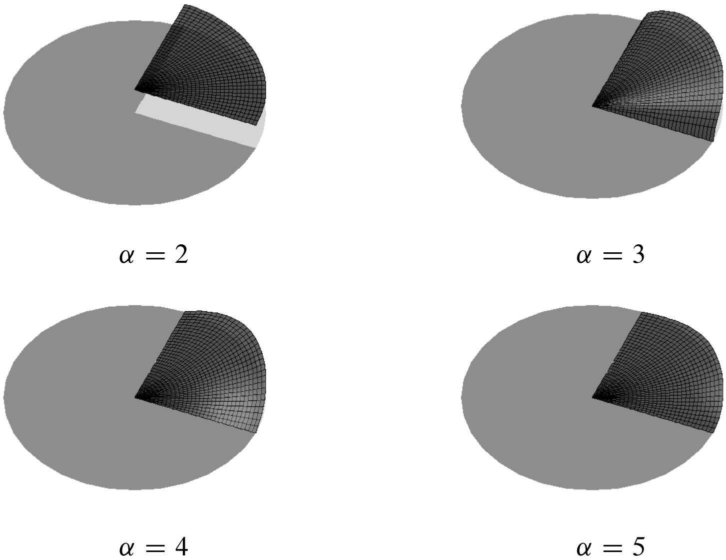


Figure 2. Graphs of Riesz distributions $R_+(\alpha)$ in two dimensions.

We now collect the important facts on Riesz distributions.

Proposition 1.2.4. *The following holds for all* $\alpha \in \mathbb{C}$*:*

(1) $\gamma \cdot R_\pm(\alpha) = \alpha(\alpha - n + 2)\, R_\pm(\alpha + 2)$.

(2) $(\mathrm{grad}\, \gamma) R_\pm(\alpha) = 2\alpha\, \mathrm{grad}\,(R_\pm(\alpha + 2))$.

(3) $\Box R_\pm(\alpha + 2) = R_\pm(\alpha)$.

(4) *For every* $\alpha \in \mathbb{C} \setminus (\{0, -2, -4, \dots\} \cup \{n-2, n-4, \dots\})$*, we have*
$$\mathrm{supp}\,(R_\pm(\alpha)) = J_\pm(0) \quad \textit{and} \quad \mathrm{sing\,supp}\,(R_\pm(\alpha)) \subset C_\pm(0).$$

(5) *For every* $\alpha \in \{0, -2, -4, \dots\} \cup \{n-2, n-4, \dots\}$*, we have* $\mathrm{supp}\,(R_\pm(\alpha)) = \mathrm{sing\,supp}\,(R_\pm(\alpha)) \subset C_\pm(0)$.

(6) *For* $n \geq 3$ *and* $\alpha = n-2, n-4, \dots, 1$ *or* 2 *respectively, we have* $\mathrm{supp}\,(R_\pm(\alpha)) = \mathrm{sing\,supp}\,(R_\pm(\alpha)) = C_\pm(0)$.

(7) $R_\pm(0) = \delta_0$.

(8) *For* $\Re e(\alpha) > 0$ *the order of* $R_\pm(\alpha)$ *is bounded from above by* $n + 1$.

(9) *If* $\alpha \in \mathbb{R}$*, then* $R_\pm(\alpha)$ *is real, i.e.,* $R_\pm(\alpha)[\varphi] \in \mathbb{R}$ *for all* $\varphi \in \mathcal{D}(V, \mathbb{R})$.

Proof. Assertions (1), (2), and (3) hold for $\Re e(\alpha) > n$ by Lemma 1.2.2. Since, after insertion of a fixed $\varphi \in \mathcal{D}(V, \mathbb{C})$, all expressions in these equations are holomorphic in α they hold for all α.

(4). Let $\varphi \in \mathcal{D}(V, \mathbb{C})$ with $\mathrm{supp}(\varphi) \cap J_\pm(0) = \emptyset$. Since $\mathrm{supp}(R_\pm(\alpha)) \subset J_\pm(0)$ for $\Re e(\alpha) > n$, it follows for those α that
$$R_\pm(\alpha)[\varphi] = 0,$$

and then for all α by Lemma 1.2.2 (4). Therefore $\mathrm{supp}(R_\pm(\alpha)) \subset J_\pm(0)$ for all α.

On the other hand, if $X \in I_\pm(0)$, then $\gamma(X) > 0$ and the map $\alpha \mapsto C(\alpha,n)\gamma(X)^{\frac{\alpha-n}{2}}$ is well defined and holomorphic on all of $\mathbb{C}$. By Lemma 1.2.2 (4) we have for $\varphi \in \mathcal{D}(V,\mathbb{C})$ with $\mathrm{supp}(\varphi) \subset I_\pm(0)$

$$R_\pm(\alpha)[\varphi] = \int_{\mathrm{supp}(\varphi)} C(\alpha,n)\gamma(X)^{\frac{\alpha-n}{2}}\varphi(X)\,dX$$

for *every* $\alpha \in \mathbb{C}$. Thus $R_\pm(\alpha)$ coincides on $I_\pm(0)$ with the smooth function $C(\alpha,n)\gamma(\cdot)^{\frac{\alpha-n}{2}}$ and therefore $\mathrm{sing\,supp}(R_\pm(\alpha)) \subset C_\pm(0)$. Since furthermore the function $\alpha \mapsto C(\alpha,n)$ vanishes only on $\{0,-2,-4,\dots\} \cup \{n-2,n-4,\dots\}$ (caused by the poles of the Gamma function), we have $I_\pm(0) \subset \mathrm{supp}(R_\pm(\alpha))$ for every $\alpha \in \mathbb{C} \setminus (\{0,-2,-4,\dots\} \cup \{n-2,n-4,\dots\})$. Thus $\mathrm{supp}(R_\pm(\alpha)) = J_\pm(0)$. This proves (4).

(5). For $\alpha \in \{0,-2,-4,\dots\} \cup \{n-2,n-4,\dots\}$ we have $C(\alpha,n) = 0$ and therefore $I_\pm(0) \cap \mathrm{supp}(R_\pm(\alpha)) = \emptyset$. Hence $\mathrm{sing\,supp}(R_\pm(\alpha)) \subset \mathrm{supp}(R_\pm(\alpha)) \subset C_\pm(0)$. It remains to show $\mathrm{supp}(R_\pm(\alpha)) \subset \mathrm{sing\,supp}(R_\pm(\alpha))$. Let $X \notin \mathrm{sing\,supp}(R_\pm(\alpha))$. Then $R_\pm(\alpha)$ coincides with a smooth function f on a neighborhood of X. Since $\mathrm{supp}(R_\pm(\alpha)) \subset C_\pm(0)$ and since $C_\pm(0)$ has a dense complement in V, we have $f \equiv 0$. Thus $X \notin \mathrm{supp}(R_\pm(\alpha))$. This proves (5).

Before we proceed to the next point we derive a more explicit formula for the Riesz distributions evaluated on testfunctions of a particular form. Introduce linear coordinates $x^1,\dots,x^n$ on V such that $\gamma(x) = -(x^1)^2 + (x^2)^2 + \dots + (x^n)^2$ and such that the x^1-axis is future directed. Let $f \in \mathcal{D}(\mathbb{R},\mathbb{C})$ and $\psi \in \mathcal{D}(\mathbb{R}^{n-1},\mathbb{C})$ and put $\varphi(x) := f(x^1)\psi(\hat{x})$ where $\hat{x} = (x^2,\dots,x^n)$. Choose the function ψ such that on $J_+(0)$ we have $\varphi(x) = f(x^1)$.

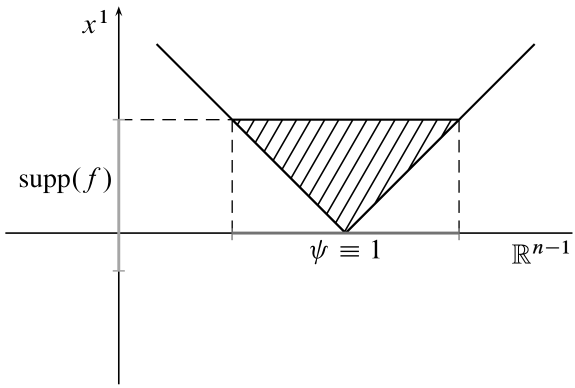


Figure 3. Support of φ.

Claim: If $\Re e(\alpha) > 1$, then

$$R_+(\alpha)[\varphi] = \frac{1}{(\alpha-1)!}\int_0^\infty r^{\alpha-1} f(r)\,dr.$$

Proof of the claim. Since both sides of the equation are holomorphic in α for $\Re e(\alpha) > 1$ it suffices to show it for $\Re e(\alpha) > n$. In that case we have by the definition of $R_+(\alpha)$

$$\begin{aligned}
R_+(\alpha)[\varphi] &= C(\alpha,n)\int_{J_+(0)} \varphi(X)\gamma(X)^{\frac{\alpha-n}{2}}\,dX \\
&= C(\alpha,n)\int_0^\infty \int_{\{|\hat{x}|<x^1\}} \varphi(x^1,\hat{x})((x^1)^2-|\hat{x}|^2)^{\frac{\alpha-n}{2}}\,d\hat{x}\,dx^1 \\
&= C(\alpha,n)\int_0^\infty f(x^1)\int_{\{|\hat{x}|<x^1\}} ((x^1)^2-|\hat{x}|^2)^{\frac{\alpha-n}{2}}\,d\hat{x}\,dx^1 \\
&= C(\alpha,n)\int_0^\infty f(x^1)\int_0^{x^1}\int_{S^{n-2}} ((x^1)^2-t^2)^{\frac{\alpha-n}{2}}t^{n-2}d\omega\,dt\,dx^1,
\end{aligned}$$

where S^{n-2} is the $(n-2)$-dimensional round sphere and $d\omega$ its standard volume element. Renaming x^1 we get

$$R_+(\alpha)[\varphi] = \mathrm{vol}(S^{n-2})\,C(\alpha,n)\int_0^\infty f(r)\int_0^r (r^2-t^2)^{\frac{\alpha-n}{2}}t^{n-2}dt\,dr.$$

Using $\int_0^r (r^2-t^2)^{\frac{\alpha-n}{2}}t^{n-2}dt = \frac{1}{2}r^{\alpha-1}\frac{(\frac{\alpha-n}{2})!(\frac{n-3}{2})!}{(\frac{\alpha-1}{2})!}$ we obtain

$$\begin{aligned}
R_+(\alpha)[\varphi] &= \frac{\mathrm{vol}(S^{n-2})}{2}\,C(\alpha,n)\int_0^\infty f(r)r^{\alpha-1}\frac{(\frac{\alpha-n}{2})!(\frac{n-3}{2})!}{(\frac{\alpha-1}{2})!}dr \\
&= \frac{1}{2}\frac{2\pi^{(n-1)/2}}{(\frac{n-1}{2}-1)!}\cdot\frac{2^{1-\alpha}\pi^{1-n/2}}{(\alpha/2-1)!(\frac{\alpha-n}{2})!}\cdot\frac{(\frac{\alpha-n}{2})!(\frac{n-3}{2})!}{(\frac{\alpha-1}{2})!}\cdot\int_0^\infty f(r)r^{\alpha-1}dr \\
&= \frac{\sqrt{\pi}\cdot 2^{1-\alpha}}{(\alpha/2-1)!(\frac{\alpha-1}{2})!}\cdot\int_0^\infty f(r)r^{\alpha-1}dr.
\end{aligned}$$

Legendre's duplication formula (see [Jeffrey1995, p. 218])

$$\left(\frac{\alpha}{2}-1\right)!\left(\frac{\alpha+1}{2}-1\right)! = 2^{1-\alpha}\sqrt{\pi}\,(\alpha-1)! \tag{1.8}$$

yields the claim.

To show (6) recall first from (5) that we know already

$$\mathrm{sing\,supp}(R_\pm(\alpha)) = \mathrm{supp}(R_\pm(\alpha)) \subset C_\pm(0)$$

for $\alpha = n-2, n-4, \ldots, 2$ or 1 respectively. Note also that the distribution $R_\pm(\alpha)$ is invariant under time-orientation preserving Lorentz transformations, that is, for any such transformation A of V we have

$$R_\pm(\alpha)[\varphi\circ A] = R_\pm(\alpha)[\varphi]$$

for every testfunction φ. Hence $\mathrm{supp}(R_\pm(\alpha))$ as well as $\mathrm{sing\,supp}(R_\pm(\alpha))$ are also invariant under the group of those transformations. Under the action of this group the orbit decomposition of $C_\pm(0)$ is given by

$$C_\pm(0) = \{0\} \cup (C_\pm(0) \setminus \{0\}).$$

Thus $\mathrm{supp}(R_\pm(\alpha)) = \mathrm{sing\,supp}(R_\pm(\alpha))$ coincides either with $\{0\}$ or with $C_\pm(0)$.

The claim shows for the testfunctions φ considered there

$$R_+(2)[\varphi] = \int_0^\infty r f(r)\, dr.$$

Hence the support of $R_+(2)$ cannot be contained in $\{0\}$. If n is even, we conclude $\mathrm{supp}(R_+(2)) = C_+(0)$ and then also $\mathrm{supp}(R_+(\alpha)) = C_+(0)$ for $\alpha = 2, 4, \ldots, n-2$.

Taking the limit $\alpha \searrow 1$ in the claim yields

$$R_+(1)[\varphi] = \int_0^\infty f(r)\, dr.$$

Now the same argument shows for odd n that $\mathrm{supp}(R_+(1)) = C_+(0)$ and then also $\mathrm{supp}(R_+(\alpha)) = C_+(0)$ for $\alpha = 1, 3, \ldots, n-2$. This concludes the proof of (6).

Proof of (7). Fix a compact subset $K \subset V$. Let $\sigma_K \in \mathcal{D}(V, \mathbb{R})$ be a function such that $\sigma_{|K} \equiv 1$. For any $\varphi \in \mathcal{D}(V, \mathbb{C})$ with $\mathrm{supp}(\varphi) \subset K$ write

$$\varphi(x) = \varphi(0) + \sum_{j=1}^n x^j \varphi_j(x)$$

with suitable smooth functions φ_j. Then

$$\begin{aligned}
R_\pm(0)[\varphi] &= R_\pm(0)[\sigma_K \varphi] \\
&= R_\pm(0)[\varphi(0)\sigma_K + \sum_{j=1}^n x^j \sigma_K \varphi_j] \\
&= \varphi(0) \underbrace{R_\pm(0)[\sigma_K]}_{=: c_K} + \sum_{j=1}^n \underbrace{(x^j R_\pm(0))}_{=0 \text{ by } (2)}[\sigma_K \varphi_j] \\
&= c_K \varphi(0).
\end{aligned}$$

The constant c_K actually does not depend on K since for $K' \supset K$ and $\mathrm{supp}(\varphi) \subset K$,

$$c_{K'}\varphi(0) = R_+(0)[\varphi] = c_K \varphi(0),$$

so that $c_K = c_{K'} =: c$. It remains to show $c = 1$.

We again look at testfunctions φ as in the claim and compute using (3)

$$\begin{aligned} c \cdot \varphi(0) &= R_+(0)[\varphi] \\ &= R_+(2)[\Box\varphi] \\ &= \int_0^\infty r f''(r)\, dr \\ &= -\int_0^\infty f'(r)\, dr \\ &= f(0) \\ &= \varphi(0). \end{aligned}$$

This concludes the proof of (7).

Proof of (8). By its definition, the distribution $R_\pm(\alpha)$ is a continuous function if $\Re e(\alpha) > n$, therefore it is of order 0. Since $\Box$ is a differential operator of order 2, the order of $\Box R_\pm(\alpha)$ is at most that of $R_\pm(\alpha)$ plus 2. It then follows from (3) that:

• If n is even: for every α with $\Re e(\alpha) > 0$ we have $\Re e(\alpha) + n = \Re e(\alpha) + 2 \cdot \frac{n}{2} > n$, so that the order of $R_\pm(\alpha)$ is not greater than n (and so $n + 1$).

• If n is odd: for every α with $\Re e(\alpha) > 0$ we have $\Re e(\alpha) + n + 1 = \Re e(\alpha) + 2 \cdot \frac{n+1}{2} > n$, so that the order of $R_\pm(\alpha)$ is not greater than $n + 1$.

This concludes the proof of (8).

Assertion (9) is clear by definition whenever $\alpha > n$. For general $\alpha \in \mathbb{R}$ choose $k \in \mathbb{N}$ so large that $\alpha + 2k > n$. Using (3) we get for any $\varphi \in \mathcal{D}(V, \mathbb{R})$

$$R_\pm(\alpha)[\varphi] = \Box^k R_\pm(\alpha + 2k)[\varphi] = R_\pm(\alpha + 2k)[\Box^k \varphi] \in \mathbb{R}$$

because $\Box^k \varphi \in \mathcal{D}(V, \mathbb{R})$ as well. □

In the following we will need a slight generalization of Lemma 1.2.2 (4):

Corollary 1.2.5. *For $\varphi \in \mathcal{D}^k(V, \mathbb{C})$ the map $\alpha \mapsto R_\pm(\alpha)[\varphi]$ defines a holomorphic function on $\left\{\alpha \in \mathbb{C} \mid \Re e(\alpha) > n - 2\left[\frac{k}{2}\right]\right\}$.*

Proof. Let $\varphi \in \mathcal{D}^k(V, \mathbb{C})$. By the definition of $R_\pm(\alpha)$ the map $\alpha \mapsto R_\pm(\alpha)[\varphi]$ is clearly holomorphic on $\{\Re e(\alpha) > n\}$. Using (3) of Proposition 1.2.4 we get the holomorphic extension to the set $\left\{\Re e(\alpha) > n - 2\left[\frac{k}{2}\right]\right\}$. □

1.3 Lorentzian geometry

We now summarize basic concepts of Lorentzian geometry. We will assume familiarity with semi-Riemannian manifolds, geodesics, the Riemannian exponential map etc. A summary of basic notions in differential geometry can be found in Appendix A.3. A thorough introduction to Lorentzian geometry can e.g. be found in [Beem–Ehrlich–Easley1996] or in [O'Neill1983]. Further results of more technical nature which could distract the reader at a first reading but which will be needed later are collected in Appendix A.5.

Let M be a time-oriented Lorentzian manifold. A piecewise C^1-curve in M is called *timelike, lightlike, causal, spacelike, future directed*, or *past directed* if its tangent vectors are timelike, lightlike, causal, spacelike, future directed, or past directed respectively. A piecewise C^1-curve in M is called *inextendible*, if no piecewise C^1-reparametrization of the curve can be continuously extended to any of the end points of the parameter interval.

The *chronological future* $I_+^M(x)$ of a point $x \in M$ is the set of points that can be reached from x by future directed timelike curves. Similarly, the *causal future* $J_+^M(x)$ of a point $x \in M$ consists of those points that can be reached from x by causal curves and of x itself. In the following, the notation $x < y$ (or $x \leq y$) will mean $y \in I_+^M(x)$ (or $y \in J_+^M(x)$ respectively). The *chronological future* of a subset $A \subset M$ is defined to be $I_+^M(A) := \bigcup_{x\in A} I_+^M(x)$. Similarly, the *causal future* of A is $J_+^M(A) := \bigcup_{x\in A} J_+^M(x)$. The *chronological past* $I_-^M(A)$ and the *causal past* $J_-^M(A)$ are defined by replacing future directed curves by past directed curves. One has in general that $I_\pm^M(A)$ is the interior of $J_\pm^M(A)$ and that $J_\pm^M(A)$ is contained in the closure of $I_\pm^M(A)$. The chronological future and past are open subsets but the causal future and past are not always closed even if A is closed (see also Section A.5 in the appendix).

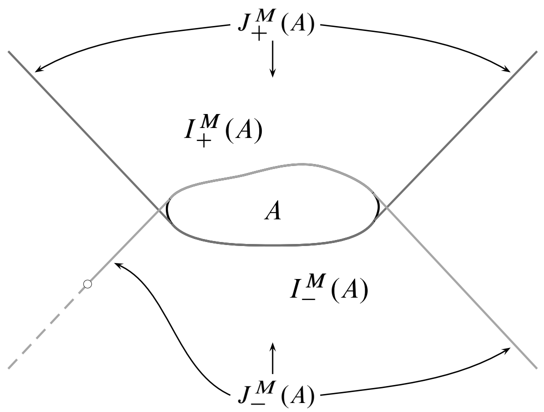


Figure 4. Causal and chronological future and past of subset A of Minkowski space with one point removed.

We will also use the notation $J^M(A) := J_-^M(A) \cup J_+^M(A)$. A subset $A \subset M$ is called *past compact* if $A \cap J_-^M(p)$ is compact for all $p \in M$. Similarly, one defines *future compact* subsets.

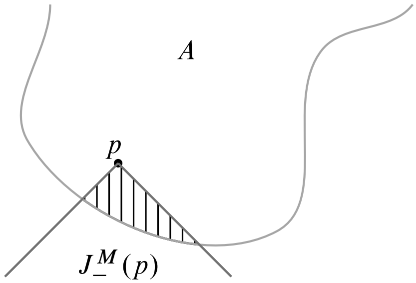


Figure 5. The subset A is past compact.

Definition 1.3.1. A subset $\Omega \subset M$ in a time-oriented Lorentzian manifold is called *causally compatible* if for all points $x \in \Omega$

$$J_\pm^\Omega(x) = J_\pm^M(x) \cap \Omega$$

holds.

Note that the inclusion "$\subset$" always holds. The condition of being causally compatible means that whenever two points in Ω can be joined by a causal curve in M this can also be done inside Ω.

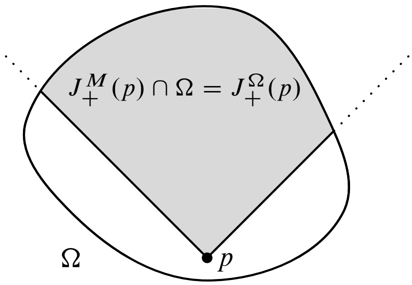


Figure 6. Causally compatible subset of Minkowski space.

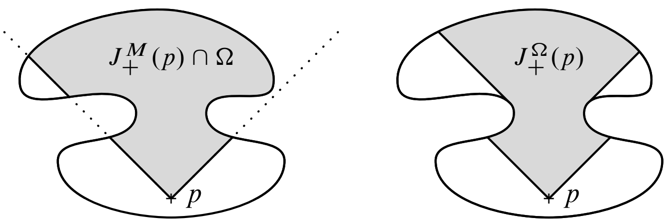


Figure 7. Domain which is not causally compatible in Minkowski space.

If $\Omega \subset M$ is a causally compatible domain in a time-oriented Lorentzian manifold, then we immediately see that for each subset $A \subset \Omega$ we have

$$J_\pm^\Omega(A) = J_\pm^M(A) \cap \Omega.$$

Note also that being causally compatible is transitive: If $\Omega \subset \Omega' \subset \Omega''$, if Ω is causally compatible in Ω', and if Ω' is causally compatible in Ω'', then so is Ω in Ω''.

Definition 1.3.2. A domain $\Omega \subset M$ in a Lorentzian manifold is called

- *geodesically starshaped* with respect to a fixed point $x \in \Omega$ if there exists an open subset $\Omega' \subset T_xM$, starshaped with respect to 0, such that the Riemannian exponential map $\exp_x$ maps Ω' diffeomorphically onto Ω;
- *geodesically convex* (or simply *convex*) if it is geodesically starshaped with respect to all of its points.

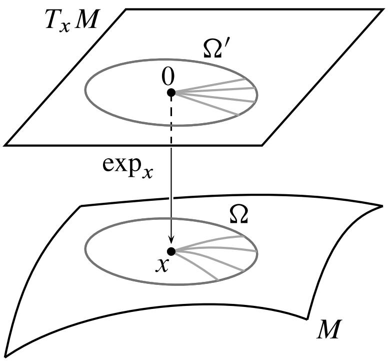


Figure 8. Ω is geodesically starshaped with respect to x.

If Ω is geodesically starshaped with respect to x, then $\exp_x(I_\pm(0) \cap \Omega') = I_\pm^\Omega(x)$ and $\exp_x(J_\pm(0) \cap \Omega') = J_\pm^\Omega(x)$. We put $C_\pm^\Omega(x) := \exp_x(C_\pm(0) \cap \Omega')$.

On a geodesically starshaped domain Ω we define the smooth positive function $\mu_x : \Omega \to \mathbb{R}$ by

$$\mathrm{dV} = \mu_x \cdot (\exp_x^{-1})^* (dz) , \tag{1.9}$$

where dV is the Lorentzian volume density and dz is the standard volume density on $T_x\Omega$. In other words, $\mu_x = \det(d\exp_x) \circ \exp_x^{-1}$. In normal coordinates about x, $\mu_x = \sqrt{|\det(g_{ij})|}$.

For each open covering of a Lorentzian manifold there exists a refinement consisting of convex open subsets, see [O'Neill1983, Chap. 5, Lemma 10].

Definition 1.3.3. A domain Ω is called *causal* if $\overline{\Omega}$ is contained in a convex domain Ω' and if for any $p, q \in \overline{\Omega}$ the intersection $J_+^{\Omega'}(p) \cap J_-^{\Omega'}(q)$ is compact and contained in $\overline{\Omega}$.

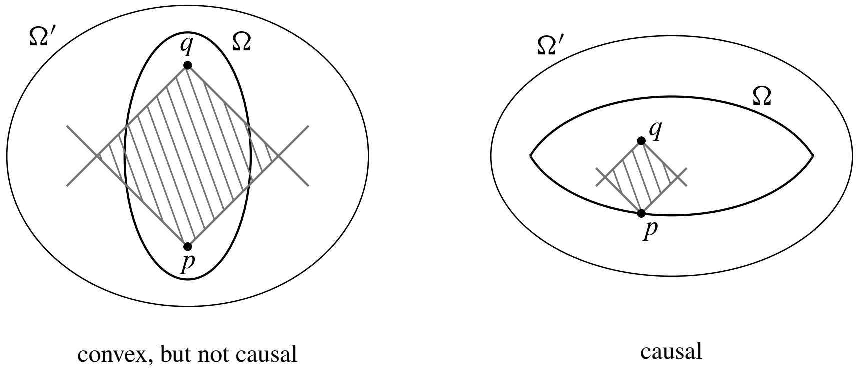


Figure 9. Convexity versus causality.

Definition 1.3.4. A subset S of a connected time-oriented Lorentzian manifold is called *achronal* (or *acausal*) if and only if each timelike (respectively causal) curve meets S at most once.

A subset S of a connected time-oriented Lorentzian manifold is a *Cauchy hypersurface* if each inextendible timelike curve in M meets S at exactly one point.

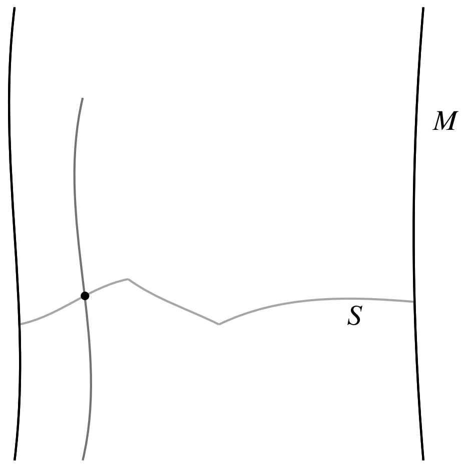


Figure 10. Cauchy hypersurface S met by a timelike curve.

Obviously every acausal subset is achronal, but the reverse is wrong. However, every achronal spacelike hypersurface is acausal (see Lemma 42 from Chap. 14 in [O'Neill1983]).

Any Cauchy hypersurface is achronal. Moreover, it is a closed topological hypersurface and it is hit by each inextendible causal curve in at least one point. Any two Cauchy hypersurfaces in M are homeomorphic. Furthermore, the causal future and past of a Cauchy hypersurface is past- and future-compact respectively. This is a consequence of e.g. [O'Neill1983, Ch. 14, Lemma 40].

Definition 1.3.5. The *Cauchy development* of a subset S of a time-oriented Lorentzian manifold M is the set $D(S)$ of points of M through which every inextendible causal curve in M meets S.

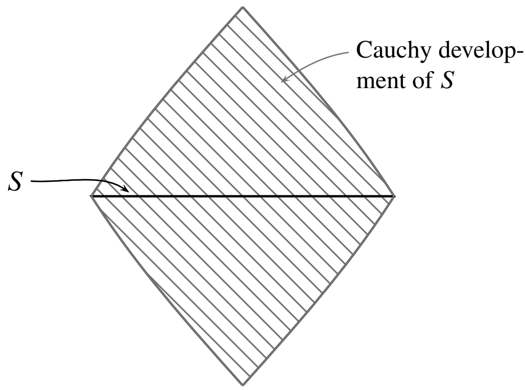


Figure 11. Cauchy development.

Remark 1.3.6. It follows from the definition that $D(D(S)) = D(S)$ for every subset $S \subset M$. Hence if $T \subset D(S)$, then $D(T) \subset D(D(S)) = D(S)$.

Of course, if S is achronal, then every inextendible causal curve in M meets S at most once. The Cauchy development $D(S)$ of every *acausal* hypersurface S is open, see [O'Neill1983, Chap. 14, Lemma 43].

Definition 1.3.7. A Lorentzian manifold is said to satisfy the *causality condition* if it does not contain any closed causal curve.

A Lorentzian manifold is said to satisfy the *strong causality condition* if there are no almost closed causal curves. More precisely, for each point $p \in M$ and for each open neighborhood U of p there exists an open neighborhood $V \subset U$ of p such that each causal curve in M starting and ending in V is entirely contained in U.

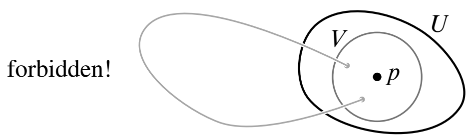


Figure 12. Strong causality condition.

Obviously, the strong causality condition implies the causality condition. Convex open subsets of a Lorentzian manifold satisfy the strong causality condition.

Definition 1.3.8. A connected time-oriented Lorentzian manifold is called *globally hyperbolic* if it satisfies the strong causality condition and if for all $p, q \in M$ the intersection $J_+^M(p) \cap J_-^M(q)$ is compact.

Remark 1.3.9. If M is a globally hyperbolic Lorentzian manifold, then a nonempty open subset $\Omega \subset M$ is itself globally hyperbolic if and only if for any $p, q \in \Omega$ the intersection $J_+^\Omega(p) \cap J_-^\Omega(q) \subset \Omega$ is compact. Indeed non-existence of almost closed causal curves in M directly implies non-existence of such curves in Ω.

We now state a very useful characterization of globally hyperbolic manifolds.

Theorem 1.3.10. *Let M be a connected time-oriented Lorentzian manifold. Then the following are equivalent:*

(1) *M is globally hyperbolic.*
(2) *There exists a Cauchy hypersurface in M.*
(3) *M is isometric to $\mathbb{R} \times S$ with metric $-\beta dt^2 + g_t$ where β is a smooth positive function, g_t is a Riemannian metric on S depending smoothly on $t \in \mathbb{R}$ and each $\{t\} \times S$ is a smooth spacelike Cauchy hypersurface in M.*

Proof. Using work of Geroch [Geroch1970, Thm. 11], it has been shown by Bernal and Sánchez in [Bernal–Sánchez2005, Thm. 1.1] that (1) implies (3). See also [Ellis–Hawking1973, Prop. 6.6.8] and [Wald1984, p. 209] for earlier mentionings of this fact. That (3) implies (2) is trivial and that (2) implies (1) is well-known, see e.g. [O'Neill1983, Cor. 39, p. 422]. □

Examples 1.3.11. Minkowski space is globally hyperbolic. Every spacelike hyperplane is a Cauchy hypersurface. One can write Minkowski space as $\mathbb{R} \times \mathbb{R}^{n-1}$ with the metric $-dt^2 + g_t$ where g_t is the Euclidean metric on $\mathbb{R}^{n-1}$ and does not depend on t.

Let (S, g_0) be a connected Riemannian manifold and $I \subset \mathbb{R}$ an interval. The manifold $M = I \times S$ with the metric $g = -dt^2 + g_0$ is globally hyperbolic if and only if (S, g_0) is complete. This applies in particular if S is compact.

More generally, if $f : I \to \mathbb{R}$ is a smooth positive function we may equip $M = I \times S$ with the metric $g = -dt^2 + f(t)^2 \cdot g_0$. Again, (M, g) is globally hyperbolic if and only if (S, g_0) is complete, see Lemma A.5.14. *Robertson–Walker spacetimes* and, in particular, *Friedmann cosmological models*, are of this type. They are used to discuss big bang, expansion of the universe, and cosmological redshift, compare [Wald1984, Ch. 5 and 6] or [O'Neill1983, Ch. 12]. Another example of this type is *deSitter spacetime*, where $I = \mathbb{R}$, $S = S^{n-1}$, g_0 is the canonical metric of S^{n-1} of constant sectional curvature 1, and $f(t) = \cosh(t)$. *Anti-deSitter spacetime* which we will discuss in more detail in Section 3.5 is not globally hyperbolic.

The interior and exterior *Schwarzschild spacetimes* are globally hyperbolic. They model the universe in the neighborhood of a massive static rotationally symmetric body such as a black hole. They are used to investigate perihelion advance of Mercury, the bending of light near the sun and other astronomical phenomena, see [Wald1984, Ch. 6] and [O'Neill1983, Ch. 13].

Corollary 1.3.12. *On every globally hyperbolic Lorentzian manifold M there exists a smooth function $h\colon M \to \mathbb{R}$ whose gradient is past directed timelike at every point and all of whose level-sets are spacelike Cauchy hypersurfaces.*

Proof. Define h to be the composition $t \circ \Phi$ where $\Phi\colon M \to \mathbb{R} \times S$ is the isometry given in Theorem 1.3.10 and $t\colon \mathbb{R} \times S \to \mathbb{R}$ is the projection onto the first factor. □

Such a function h on a globally hyperbolic Lorentzian manifold will be referred to as a *Cauchy time-function*. Note that a Cauchy time-function is strictly monotonically increasing along any future directed causal curve.

We quote an enhanced form of Theorem 1.3.10, due to A. Bernal and M. Sánchez (see [Bernal–Sánchez2006, Theorem 1.2]), which will be needed in Chapter 3.

Theorem 1.3.13. *Let M be a globally hyperbolic manifold and S be a spacelike smooth Cauchy hypersurface in M. Then there exists a Cauchy time-function $h\colon M \to \mathbb{R}$ such that $S = h^{-1}(\{0\})$.* □

Any given smooth spacelike Cauchy hypersurface in a (necessarily globally hyperbolic) Lorentzian manifold is therefore the leaf of a foliation by smooth spacelike Cauchy hypersurfaces.

Recall that the *length* $L[c]$ of a piecewise C^1-curve $c\colon [a,b] \to M$ on a Lorentzian manifold (M,g) is defined by

$$L[c] := \int_a^b \sqrt{|g(\dot c(t), \dot c(t))|}\,dt.$$

Definition 1.3.14. The *time-separation* on a Lorentzian manifold (M,g) is the function $\tau\colon M \times M \to \mathbb{R} \cup \{\infty\}$ defined by

$$\tau(p,q) := \begin{cases} \sup\{L[c] \mid c \text{ future directed causal curve from } p \text{ to } q, & \text{if } p < q \\ 0, & \text{otherwise,} \end{cases}$$

for all p, q in M.

The properties of τ which will be needed later are the following:

Proposition 1.3.15. *Let M be a time-oriented Lorentzian manifold. Let p, q, and $r \in M$. Then*

(1) $\tau(p,q) > 0$ *if and only if* $q \in I_+^M(p)$.
(2) *The function τ is lower semi-continuous on $M \times M$. If M is convex or globally hyperbolic, then τ is finite and continuous.*

(3) *The function* τ *satisfies the* inverse triangle inequality*: If* $p \leq q \leq r$*, then*

$$\tau(p,r) \geq \tau(p,q) + \tau(q,r). \tag{1.10}$$

See e.g. Lemmas 16, 17, and 21 from Chapter 14 in [O'Neill1983] for a proof. □

Now let M be a Lorentzian manifold. For a differentiable function $f : M \to \mathbb{R}$, the *gradient* of f is the vector field

$$\operatorname{grad} f := (df)^{\sharp}. \tag{1.11}$$

Here $\omega \mapsto \omega^{\sharp}$ denotes the canonical isomorphism $T^*M \to TM$ induced by the Lorentzian metric, i.e., for $\omega \in T_x^*M$ the vector $\omega^{\sharp} \in T_xM$ is characterized by the fact that $\omega(X) = \langle \omega^{\sharp}, X \rangle$ for all $X \in T_xM$. The inverse isomorphism $TM \to T^*M$ is denoted by $X \mapsto X^{\flat}$. One easily checks that for differentiable functions $f, g : M \to \mathbb{R}$

$$\operatorname{grad}(fg) = g \operatorname{grad} f + f \operatorname{grad} g. \tag{1.12}$$

Locally, the gradient of f can be written as

$$\operatorname{grad} f = \sum_{j=1}^{n} \varepsilon_j \, df(e_j) \, e_j$$

where $e_1, \dots, e_n$ is a local Lorentz orthonormal frame of TM, $\varepsilon_j = \langle e_j, e_j \rangle = \pm 1$. For a differentiable vector field X on M the *divergence* is the function

$$\operatorname{div} X := \operatorname{tr}(\nabla X) = \sum_{j=1}^{n} \varepsilon_j \langle e_j, \nabla_{e_j} X \rangle.$$

If X is a differentiable vector field and f a differentiable function on M, then one immediately sees that

$$\operatorname{div}(fX) = f \operatorname{div} X + \langle \operatorname{grad} f, X \rangle. \tag{1.13}$$

There is another way to characterize the divergence. Let dV be the volume form induced by the Lorentzian metric. Inserting the vector field X yields an $(n-1)$-form $\mathrm{dV}(X, \cdot, \dots, \cdot)$. Hence $d(\mathrm{dV}(X, \cdot, \dots, \cdot))$ is an n-form and can therefore be written as a function times dV, namely

$$d(\mathrm{dV}(X, \cdot, \dots, \cdot)) = \operatorname{div} X \cdot \mathrm{dV}. \tag{1.14}$$

This shows that the divergence operator depends only mildly on the Lorentzian metric. If two Lorentzian (or more generally, semi-Riemannian) metrics have the same volume form, then they also have the same divergence operator. This is certainly not true for the gradient.

The divergence is important because of Gauss' divergence theorem:

Theorem 1.3.16. *Let M be a Lorentzian manifold and let $D \subset M$ be a domain with piecewise smooth boundary. We assume that the induced metric on the smooth part of the boundary is non-degenerate, i.e., it is either Riemannian or Lorentzian on each connected component. Let $\mathfrak{n}$ denote the exterior normal field along ∂D, normalized to $\langle \mathfrak{n}, \mathfrak{n} \rangle =: \varepsilon_{\mathfrak{n}} = \pm 1$.*

Then for every smooth vector field X on M such that $\mathrm{supp}(X) \cap \overline{D}$ *is compact we have*

$$\int_D \mathrm{div}(X)\ \mathrm{dV} = \int_{\partial D} \varepsilon_{\mathfrak{n}} \langle X, \mathfrak{n} \rangle\, \mathrm{dA} \qquad \square$$

where dA *is the induced volume element on ∂D.*

Let $e_1, \ldots, e_n$ be a Lorentz orthonormal basis of $T_x M$. Then $(\xi^1, \ldots, \xi^n) \mapsto \exp_x(\sum_j \xi^j e_j)$ is a local diffeomorphism of a neighborhood of 0 in $\mathbb{R}^n$ onto a neighborhood of x in M. This defines coordinates $\xi^1, \ldots, \xi^n$ on any open neighborhood of x which is geodesically starshaped with respect to x. Such coordinates are called *normal coordinates* about the point x.

We express the vector X in normal coordinates about x and write $X = \sum_j \eta^j \frac{\partial}{\partial \xi^j}$. From (1.14) we conclude, using $\mathrm{dV} = \mu_x \cdot d\xi^1 \wedge \cdots \wedge d\xi^n$

$$\begin{aligned}
\mathrm{div}(\mu_x^{-1} X) \cdot \mathrm{dV} &= d(\,\mathrm{dV}(\mu_x^{-1} X, \cdot, \ldots, \cdot)) \\
&= d\Big(\sum_j (-1)^{j-1}\, \eta^j\, d\xi^1 \wedge \cdots \wedge \widehat{d\xi^j} \wedge \cdots \wedge d\xi^n \Big) \\
&= \sum_j (-1)^{j-1}\, d\eta^j \wedge d\xi^1 \wedge \cdots \wedge \widehat{d\xi^j} \wedge \cdots \wedge d\xi^n \\
&= \sum_j \frac{\partial \eta^j}{\partial \xi^j}\, d\xi^1 \wedge \cdots \wedge d\xi^n \\
&= \sum_j \frac{\partial \eta^j}{\partial \xi^j}\, \mu_x^{-1}\ \mathrm{dV}.
\end{aligned}$$

Thus

$$\mu_x\ \mathrm{div}(\mu_x^{-1} X) = \sum_j \frac{\partial \eta^j}{\partial \xi^j}. \tag{1.15}$$

For a C^2-function f the *Hessian* at x is the symmetric bilinear form

$$\mathrm{Hess}(f)|_x \colon T_x M \times T_x M \to \mathbb{R}, \quad \mathrm{Hess}(f)|_x(X, Y) := \langle \nabla_X \operatorname{grad} f, Y \rangle.$$

The *d'Alembert operator* is defined by

$$\Box f := -\operatorname{tr}(\mathrm{Hess}(f)) = -\operatorname{div}\operatorname{grad} f.$$

If $f \colon M \to \mathbb{R}$ and $F \colon \mathbb{R} \to \mathbb{R}$ are C^2 a straightforward computation yields

$$\Box(F \circ f) = -(F'' \circ f)\langle df, df \rangle + (F' \circ f)\Box f. \tag{1.16}$$

Lemma 1.3.17. *Let Ω be a domain in M, geodesically starshaped with respect to $x \in \Omega$. Then the function μ_x defined in (1.9) satisfies*

$$\mu_x(x) = 1, \quad d\mu_x|_x = 0, \quad \mathrm{Hess}(\mu_x)|_x = -\frac{1}{3}\,\mathrm{ric}_x, \quad (\Box\mu_x)(x) = \frac{1}{3}\mathrm{scal}(x),$$

where ric_x *denotes the Ricci curvature considered as a bilinear form on $T_x\Omega$ and* scal *is the scalar curvature.*

Proof. Let $X \in T_x\Omega$ be fixed. Let $e_1, \dots, e_n$ be a Lorentz orthonormal basis of $T_x\Omega$. Denote by $J_1, \dots, J_n$ the Jacobi fields along $c(t) = \exp_x(tX)$ satisfying $J_j(0) = 0$ and $\frac{\nabla J_j}{dt}(0) = e_j$ for every $1 \le j \le n$. The differential of $\exp_x$ at tX is, for every t for which it is defined, given by

$$d_{tX}\exp_x(e_j) = \frac{1}{t}J_j(t),$$

$j = 1, \dots, n$. From the definition of μ_x we have

$$\begin{aligned}\mu_x(\exp_x(tX))e_1 \wedge \cdots \wedge e_n &= \det(d_{tX}\exp_x)e_1 \wedge \cdots \wedge e_n \\ &= (d_{tX}\exp_x(e_1)) \wedge \cdots \wedge (d_{tX}\exp_x(e_n)) \\ &= \frac{1}{t}J_1(t) \wedge \cdots \wedge \frac{1}{t}J_n(t).\end{aligned}$$

Jacobi fields J along the geodesic $c(t) = \exp_x(tX)$ satisfy the Jacobi field equation $\frac{\nabla^2}{dt^2}J(t) = -R(J(t), \dot c(t))\dot c(t)$, where R denotes the curvature tensor of the Levi-Civita connection ∇. Differentiating this once more yields $\frac{\nabla^3}{dt^3}J(t) = -\frac{\nabla R}{dt}(J(t), \dot c(t))\dot c(t) - R(\frac{\nabla}{dt}J(t), \dot c(t))\dot c(t)$. For $J = J_j$ and $t = 0$ we have $J_j(0) = 0$, $\frac{\nabla J_j}{dt}(0) = e_j$, $\frac{\nabla^2 J_j}{dt^2}(0) = -R(0, \dot c(0))\dot c(0) = 0$, and $\frac{\nabla^3 J_j}{dt^3}(0) = -R(e_j, X)X$ where $X = \dot c(0)$. Identifying $J_j(t)$ with its parallel translate to $T_x\Omega$ along c the Taylor expansion of J_j up to order 3 reads as

$$J_j(t) = te_j - \frac{t^3}{6}R(e_j, X)X + \mathrm{O}(t^4).$$

This implies

$$\begin{aligned}\frac{1}{t}J_1(t) \wedge \cdots \wedge \frac{1}{t}J_n(t) &= e_1 \wedge \cdots \wedge e_n \\ &\qquad - \frac{t^2}{6}\sum_{j=1}^n e_1 \wedge \cdots \wedge R(e_j, X)X \wedge \cdots \wedge e_n + \mathrm{O}(t^3) \\ &= e_1 \wedge \cdots \wedge e_n \\ &\qquad - \frac{t^2}{6}\sum_{j=1}^n \varepsilon_j \langle R(e_j, X)X, e_j\rangle e_1 \wedge \cdots \wedge e_n + \mathrm{O}(t^3) \\ &= \Big(1 - \frac{t^2}{6}\,\mathrm{ric}(X, X) + \mathrm{O}(t^3)\Big)e_1 \wedge \cdots \wedge e_n.\end{aligned}$$

Thus

$$\mu_x(\exp_x(tX)) = 1 - \frac{t^2}{6}\,\mathrm{ric}(X,X) + \mathrm{O}(t^3)$$

and therefore

$$\mu_x(x) = 1, \quad d\mu_x(X) = 0, \quad \mathrm{Hess}(\mu_x)(X,X) = -\frac{1}{3}\,\mathrm{ric}(X,X).$$

Taking a trace yields the result for the d'Alembertian. □

Lemma 1.3.17 and (1.16) with $f = \mu_x$ and $F(t) = t^{-1/2}$ yield:

Corollary 1.3.18. *Under the assumptions of Lemma 1.3.17 one has*

$$(\Box\mu_x^{-1/2})(x) = -\frac{1}{6}\mathrm{scal}(x). \qquad \Box$$

Let Ω be a domain in a Lorentzian manifold M, geodesically starshaped with respect to $x \in \Omega$. Set

$$\Gamma_x := \gamma \circ \exp_x^{-1} : \Omega \to \mathbb{R} \tag{1.17}$$

where γ is defined as in (1.6) with $V = T_x\Omega$.

Lemma 1.3.19. *Let M be a time-oriented Lorentzian manifold. Let the domain $\Omega \subset M$ be geodesically starshaped with respect to $x \in \Omega$. Then the following holds on Ω:*

(1) $\langle \mathrm{grad}\,\Gamma_x, \mathrm{grad}\,\Gamma_x\rangle = -4\Gamma_x$.

(2) *On $I_+^\Omega(x)$ (or on $I_-^\Omega(x)$) the gradient* $\mathrm{grad}\,\Gamma_x$ *is a past directed (or future directed respectively) timelike vector field.*

(3) $\Box\Gamma_x - 2n = -\langle \mathrm{grad}\,\Gamma_x, \mathrm{grad}(\log(\mu_x))\rangle$.

Proof. (1) Let $y \in \Omega$ and $Z \in T_y\Omega$. The differential of γ at a point p is given by $d_p\gamma = -2\langle p, \cdot\rangle$. Hence

$$\begin{aligned} d_y\Gamma_x(Z) &= d_{\exp_x^{-1}(y)}\gamma \circ d_y \exp_x^{-1}(Z) \\ &= -2\langle \exp_x^{-1}(y), d_y \exp_x^{-1}(Z)\rangle. \end{aligned}$$

Applying the Gauss Lemma [O'Neill1983, p. 127], we obtain

$$d_y\Gamma_x(Z) = -2\langle d_{\exp_x^{-1}(y)} \exp_x(\exp_x^{-1}(y)), Z\rangle.$$

Thus

$$\mathrm{grad}_y\,\Gamma_x = -2 d_{\exp_x^{-1}(y)} \exp_x(\exp_x^{-1}(y)). \tag{1.18}$$

It follows again from the Gauss Lemma that

$$\begin{aligned} \langle \mathrm{grad}_y\,\Gamma_x, \mathrm{grad}_y\,\Gamma_x\rangle &= 4\langle d_{\exp_x^{-1}(y)} \exp_x(\exp_x^{-1}(y)), d_{\exp_x^{-1}(y)} \exp_x(\exp_x^{-1}(y))\rangle \\ &= 4\langle \exp_x^{-1}(y), \exp_x^{-1}(y)\rangle \\ &= -4\Gamma_x(y). \end{aligned}$$

(2) On $I_+^\Omega(x)$ the function Γ_x is positive, hence $\langle \operatorname{grad} \Gamma_x, \operatorname{grad} \Gamma_x \rangle = -4\Gamma_x < 0$. Thus $\operatorname{grad} \Gamma_x$ is timelike. For a future directed timelike tangent vector $Z \in T_x\Omega$ the curve $c(t) := \exp_x(tZ)$ is future directed timelike and Γ_x increases along c. Hence $0 \leq \frac{d}{dt}(\Gamma_x \circ c) = \langle \operatorname{grad} \Gamma_x, \dot{c} \rangle$. Thus $\operatorname{grad} \Gamma_x$ is past directed along c. Since every point in $I_+^\Omega(x)$ can be written in the form $\exp_x(Z)$ for a future directed timelike tangent vector Z this proves the assertion for $I_+^\Omega(x)$. The argument for $I_-^\Omega(x)$ is analogous.

(3) Using (1.13) with $f = \mu_x^{-1}$ and $X = \operatorname{grad} \Gamma_x$ we get

$$\operatorname{div}(\mu_x^{-1} \operatorname{grad} \Gamma_x) = \mu_x^{-1} \operatorname{div} \operatorname{grad} \Gamma_x + \langle \operatorname{grad}(\mu_x^{-1}), \operatorname{grad} \Gamma_x \rangle$$

and therefore

$$\begin{aligned} \Box \Gamma_x &= \langle \operatorname{grad}(\log(\mu_x^{-1})), \operatorname{grad} \Gamma_x \rangle - \mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad} \Gamma_x) \\ &= -\langle \operatorname{grad}(\log(\mu_x)), \operatorname{grad} \Gamma_x \rangle - \mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad} \Gamma_x). \end{aligned}$$

It remains to show $\mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad} \Gamma_x) = -2n$. We check this in normal coordinates $\xi^1, \dots, \xi^n$ about x. By (1.18) we have $\operatorname{grad} \Gamma_x = -2 \sum_j \xi^j \frac{\partial}{\partial \xi^j}$ so that (1.15) implies

$$\mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad} \Gamma_x) = -2 \sum_j \frac{\partial \xi^j}{\partial \xi^j} = -2n. \qquad \Box$$

Remark 1.3.20. If Ω is convex and τ is the time-separation function of Ω, then one can check that

$$\tau(p,q) = \begin{cases} \sqrt{\Gamma(p,q)}, & \text{if } p < q \\ 0, & \text{otherwise.} \end{cases}$$

1.4 Riesz distributions on a domain

Riesz distributions have been defined on all spaces isometric to Minkowski space. They are therefore defined on the tangent spaces at all points of a Lorentzian manifold. We now show how to construct Riesz distributions defined in small open subsets of the Lorentzian manifold itself. The passage from the tangent space to the manifold will be provided by the Riemannian exponential map.

Let Ω be a domain in a time-oriented n-dimensional Lorentzian manifold, $n \geq 2$. Suppose Ω is geodesically starshaped with respect to some point $x \in \Omega$. In particular, the Riemannian exponential function $\exp_x$ is a diffeomorphism from $\Omega' := \exp^{-1}(\Omega) \subset T_x\Omega$ to Ω. Let $\mu_x \colon \Omega \to \mathbb{R}$ be defined as in (1.9). Put

$$R_\pm^\Omega(\alpha, x) := \mu_x \exp_x^* R_\pm(\alpha),$$

that is, for every testfunction $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$,

$$R_\pm^\Omega(\alpha, x)[\varphi] := R_\pm(\alpha)[(\mu_x \varphi) \circ \exp_x].$$

Note that $\mathrm{supp}((\mu_x\varphi)\circ\exp_x)$ is contained in Ω'. Extending the function $(\mu_x\varphi)\circ\exp_x$ by zero we can regard it as a testfunction on $T_x\Omega$ and thus apply $R_\pm(\alpha)$ to it.

Definition 1.4.1. We call $R_+^\Omega(\alpha, x)$ the *advanced Riesz distribution* and $R_-^\Omega(\alpha, x)$ the *retarded Riesz distribution* on Ω at x for $\alpha \in \mathbb{C}$.

The relevant properties of the Riesz distributions are collected in the following proposition.

Proposition 1.4.2. *The following holds for all $\alpha \in \mathbb{C}$ and all $x \in \Omega$:*

(1) *If $\Re\mathrm{e}(\alpha) > n$, then $R_\pm^\Omega(\alpha, x)$ is the continuous function*

$$R_\pm^\Omega(\alpha, x) = \begin{cases} C(\alpha, n)\,\Gamma_x^{\frac{\alpha-n}{2}} & \text{on } J_\pm^\Omega(x), \\ 0 & \text{elsewhere.} \end{cases}$$

(2) *For every fixed testfunction φ the map $\alpha \mapsto R_\pm^\Omega(\alpha, x)[\varphi]$ is holomorphic on $\mathbb{C}$.*

(3) $\Gamma_x \cdot R_\pm^\Omega(\alpha, x) = \alpha(\alpha - n + 2)\, R_\pm^\Omega(\alpha + 2, x)$.

(4) $\mathrm{grad}\,(\Gamma_x) \cdot R_\pm^\Omega(\alpha, x) = 2\alpha\, \mathrm{grad}\, R_\pm^\Omega(\alpha + 2, x)$.

(5) *If $\alpha \neq 0$, then $\Box R_\pm^\Omega(\alpha + 2, x) = \big(\frac{\Box\Gamma_x - 2n}{2\alpha} + 1\big) R_\pm^\Omega(\alpha, x)$.*

(6) $R_\pm^\Omega(0, x) = \delta_x$.

(7) *For every $\alpha \in \mathbb{C} \setminus (\{0, -2, -4, \dots\} \cup \{n-2, n-4, \dots\})$ we have*

$$\mathrm{supp}\,\big(R_\pm^\Omega(\alpha, x)\big) = J_\pm^\Omega(x) \quad \text{and} \quad \mathrm{sing\,supp}\,\big(R_\pm^\Omega(\alpha, x)\big) \subset C_\pm^\Omega(x).$$

(8) *For every $\alpha \in \{0, -2, -4, \dots\} \cup \{n-2, n-4, \dots\}$ we have*

$$\mathrm{supp}\,\big(R_\pm^\Omega(\alpha, x)\big) = \mathrm{sing\,supp}\,\big(R_\pm^\Omega(\alpha, x)\big) \subset C_\pm^\Omega(x).$$

(9) *For $n \geq 3$ and $\alpha = n-2, n-4, \dots, 1$ or 2 respectively we have*

$$\mathrm{supp}\,\big(R_\pm^\Omega(\alpha, x)\big) = \mathrm{sing\,supp}\,\big(R_\pm^\Omega(\alpha, x)\big) = C_\pm^\Omega(x).$$

(10) *For $\Re\mathrm{e}(\alpha) > 0$ we have $\mathrm{ord}(R_\pm^\Omega(\alpha, x)) \leq n + 1$. Moreover, there exists a neighborhood U of x and a constant $C > 0$ such that*

$$|R_\pm^\Omega(\alpha, x')[\varphi]| \leq C \cdot \|\varphi\|_{C^{n+1}(\Omega)}$$

for all $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$ and all $x' \in U$.

(11) *If $U \subset \Omega$ is an open neighborhood of x such that Ω is geodesically starshaped with respect to all $x' \in U$ and if $V \in \mathcal{D}(U \times \Omega, \mathbb{C})$, then the function $U \to \mathbb{C}$, $x' \mapsto R_\pm^\Omega(\alpha, x')[y \mapsto V(x', y)]$, is smooth.*

(12) *If $U \subset \Omega$ is an open neighborhood of x such that Ω is geodesically starshaped with respect to all $x' \in U$, if $\Re\mathrm{e}(\alpha) > 0$, and if $V \in \mathcal{D}^{n+1+k}(U \times \Omega, \mathbb{C})$, then the function $U \to \mathbb{C}$, $x' \mapsto R_\pm^\Omega(\alpha, x')[y \mapsto V(x', y)]$, is C^k.*

(13) *For every* $\varphi \in \mathcal{D}^k(\Omega, \mathbb{C})$ *the map* $\alpha \mapsto R_\pm^\Omega(\alpha, x)[\varphi]$ *is a holomorphic function on* $\{\alpha \in \mathbb{C} \mid \Re e(\alpha) > n - 2[\frac{k}{2}]\}$.

(14) *If* $\alpha \in \mathbb{R}$, *then* $R_\pm^\Omega(\alpha, x)$ *is real, i.e.,* $R_\pm^\Omega(\alpha, x)[\varphi] \in \mathbb{R}$ *for all* $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$.

Proof. It suffices to prove the statements for the advanced Riesz distributions.

(1) Let $\Re e(\alpha) > n$ and $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$. Then

$$\begin{aligned} R_+^\Omega(\alpha, x)[\varphi] &= R_+^\Omega(\alpha, x)[(\mu_x \circ \exp_x) \cdot (\varphi \circ \exp_x)] \\ &= C(\alpha, n) \int_{J_+(0)} \gamma^{\frac{\alpha-n}{2}} \cdot (\varphi \circ \exp_x) \cdot \mu_x \, dz \\ &= C(\alpha, n) \int_{J_+^\Omega(x)} \Gamma_x^{\frac{\alpha-n}{2}} \cdot \varphi \, \mathrm{dV}. \end{aligned}$$

(2) This follows directly from the definition of $R_+^\Omega(\alpha, x)$ and from Lemma 1.2.2 (4).

(3) By (1) this obviously holds for $\Re e(\alpha) > n$ since $C(\alpha, n) = \alpha(\alpha - n + 2) \cdot C(\alpha + 2, n)$. By analyticity of $\alpha \mapsto R_+^\Omega(\alpha, x)$ it must hold for all α.

(4) Consider α with $\Re e(\alpha) > n$. By (1) the function $R_+^\Omega(\alpha + 2, x)$ is then C^1. On $J_+^\Omega(x)$ we compute

$$\begin{aligned} 2\alpha \,\mathrm{grad}\, R_+^\Omega(\alpha + 2, x) &= 2\alpha C(\alpha + 2, n) \,\mathrm{grad} \left(\Gamma_x^{\frac{\alpha+2-n}{2}} \right) \\ &= \underbrace{2\alpha C(\alpha + 2, n) \frac{\alpha + 2 - n}{2}}_{C(\alpha,n)} \Gamma_x^{\frac{\alpha-n}{2}} \,\mathrm{grad}\, \Gamma_x \\ &= R_+^\Omega(\alpha, x) \,\mathrm{grad}\, \Gamma_x. \end{aligned}$$

For arbitrary $\alpha \in \mathbb{C}$ assertion (4) follows from analyticity of $\alpha \mapsto R_+^\Omega(\alpha, x)$.

(5) Let $\alpha \in \mathbb{C}$ with $\Re e(\alpha) > n+2$. Since $R_+^\Omega(\alpha+2, x)$ is then C^2, we can compute $\Box R_+^\Omega(\alpha+2, x)$ classically. This will show that (5) holds for all α with $\Re e(\alpha) > n+2$. Analyticity then implies (5) for all α.

$$\begin{aligned} &\Box R_+^\Omega(\alpha + 2, x) \\ &\quad = -\mathrm{div} \left(\mathrm{grad}\, R_+^\Omega(\alpha + 2, x) \right) \\ &\quad \overset{(4)}{=} -\frac{1}{2\alpha} \,\mathrm{div} \left(R_+^\Omega(\alpha, x) \cdot \mathrm{grad}(\Gamma_x) \right) \\ &\quad \overset{(1.13)}{=} \frac{1}{2\alpha} \Box \Gamma_x \cdot R_+^\Omega(\alpha, x) - \frac{1}{2\alpha} \langle \mathrm{grad}\, \Gamma_x, \mathrm{grad}\, R_+^\Omega(\alpha, x) \rangle \\ &\quad \overset{(4)}{=} \frac{1}{2\alpha} \Box \Gamma_x \cdot R_+^\Omega(\alpha, x) - \frac{1}{2\alpha \cdot 2(\alpha - 2)} \langle \mathrm{grad}\, \Gamma_x, \mathrm{grad}\, \Gamma_x \cdot R_+^\Omega(\alpha - 2, x) \rangle \\ &\quad \overset{\text{Lemma 1.3.19(1)}}{=} \frac{1}{2\alpha} \Box \Gamma_x \cdot R_+^\Omega(\alpha, x) + \frac{1}{\alpha(\alpha - 2)} \Gamma_x \cdot R_+^\Omega(\alpha - 2, x) \end{aligned}$$

$$\overset{(3)}{=} \frac{1}{2\alpha}\,\Box\Gamma_x \cdot R_+^\Omega(\alpha,x) + \frac{(\alpha-2)(\alpha-n)}{\alpha(\alpha-2)}\,R_+^\Omega(\alpha,x)$$
$$= \left(\frac{\Box\Gamma_x - 2n}{2\alpha} + 1\right) R_+^\Omega(\alpha,x).$$

(6) Let φ be a testfunction on Ω. Then by Proposition 1.2.4 (7)

$$\begin{aligned} R_+^\Omega(0,x)[\varphi] &= R_+(0)[(\mu_x\varphi)\circ \exp_x] \\ &= \delta_0[(\mu_x\varphi)\circ \exp_x] \\ &= ((\mu_x\varphi)\circ \exp_x)(0) \\ &= \mu_x(x)\varphi(x) \\ &= \varphi(x) \\ &= \delta_x[\varphi]. \end{aligned}$$

(11) Let $A(x,x')\colon T_x\Omega \to T_{x'}\Omega$ be a time-orientation preserving linear isometry. Then

$$R_+^\Omega(\alpha,x')[V(x',\cdot)] = R_+(\alpha)[(\mu_{x'}\cdot V(x',\cdot))\circ \exp_{x'}\circ A(x,x')]$$

where $R_+(\alpha)$ is, as before, the Riesz distribution on $T_x\Omega$. Hence if we choose $A(x,x')$ to depend smoothly on x', then $(\mu_{x'}\cdot V(x',y))\circ \exp_{x'}\circ A(x,x')$ is smooth in x' and y and the assertion follows from Lemma 1.1.6.

(10) Since $\mathrm{ord}(R_\pm(\alpha)) \le n+1$ by Proposition 1.2.4 (8) we have $\mathrm{ord}(R_\pm^\Omega(\alpha,x)) \le n+1$ as well. From the definition $R_\pm^\Omega(\alpha,x) = \mu_x \exp_x^* R_\pm(\alpha)$ it is clear that the constant C may be chosen locally uniformly in x.

(12) By (10) we can apply $R_\pm^\Omega(\alpha,x')$ to $V(x',\cdot)$. Now the same argument as for (11) shows that the assertion follows from Lemma 1.1.6.

The remaining assertions follow directly from the corresponding properties of the Riesz distributions on Minkowski space. For example (13) is a consequence of Corollary 1.2.5. $\Box$

Advanced and retarded Riesz distributions are related as follows.

Lemma 1.4.3. *Let Ω be a convex time-oriented Lorentzian manifold. Let $\alpha \in \mathbb{C}$. Then for all $u \in \mathcal{D}(\Omega\times\Omega,\mathbb{C})$ we have*

$$\int_\Omega R_+^\Omega(\alpha,x)\,[y\mapsto u(x,y)]\;\mathrm{dV}(x) = \int_\Omega R_-^\Omega(\alpha,y)\,[x\mapsto u(x,y)]\;\mathrm{dV}(y).$$

Proof. The convexity condition for Ω ensures that the Riesz distributions $R_\pm^\Omega(\alpha,x)$ are defined for all $x\in\Omega$. By Proposition 1.4.2 (11) the integrands are smooth. Since u has compact support contained in $\Omega\times\Omega$ the integrand $R_+^\Omega(\alpha,x)\,[y\mapsto u(x,y)]$ (as a function in x) has compact support contained in Ω. A similar statement holds for the integrand of the right-hand side. Hence the integrals exist. By Proposition 1.4.2 (13) they are holomorphic in α. Thus it suffices to show the equation for α with $\mathfrak{Re}(\alpha) > n$.

For such an $\alpha \in \mathbb{C}$ the Riesz distributions $R_+(\alpha, x)$ and $R_-(\alpha, y)$ are continuous functions. From the explicit formula (1) in Proposition 1.4.2 we see

$$R_+(\alpha, x)(y) = R_-(\alpha, y)(x)$$

for all $x, y \in \Omega$. By Fubini's theorem we get

$$\begin{aligned}\int_\Omega R_+^\Omega(\alpha, x)[y \mapsto u(x, y)]\, \mathrm{dV}(x) &= \int_\Omega \left(\int_\Omega R_+^\Omega(\alpha, x)(y)\, u(x, y)\, \mathrm{dV}(y) \right) \mathrm{dV}(x) \\ &= \int_\Omega \left(\int_\Omega R_-^\Omega(\alpha, y)(x)\, u(x, y)\, \mathrm{dV}(x) \right) \mathrm{dV}(y) \\ &= \int_\Omega R_-^\Omega(\alpha, y)[x \mapsto u(x, y)]\, \mathrm{dV}(y). \qquad \square\end{aligned}$$

As a technical tool we will also need a version of Lemma 1.4.3 for certain nonsmooth sections.

Lemma 1.4.4. *Let Ω be a causal domain in a time-oriented Lorentzian manifold of dimension n. Let $\Re\mathrm{e}(\alpha) > 0$ and let $k \geq n + 1$. Let K_1, K_2 be compact subsets of $\overline{\Omega}$ and let $u \in C^k(\overline{\Omega} \times \overline{\Omega})$ so that $\mathrm{supp}(u) \subset J_+^\Omega(K_1) \times J_-^\Omega(K_2)$. Then*

$$\int_\Omega R_+^\Omega(\alpha, x)\, [y \mapsto u(x, y)]\, \mathrm{dV}(x) = \int_\Omega R_-^\Omega(\alpha, y)\, [x \mapsto u(x, y)]\, \mathrm{dV}(y).$$

Proof. For fixed x, the support of the function $y \mapsto u(x, y)$ is contained in $J_-^\Omega(K_2)$. Since Ω is causal, it follows from Lemma A.5.3 that the subset $J_-^\Omega(K_2) \cap J_+^\Omega(x)$ is relatively compact in $\overline{\Omega}$. Therefore the intersection of the supports of $y \mapsto u(x, y)$ and $R_+^\Omega(\alpha, x)$ is compact and contained in $\overline{\Omega}$. By Proposition 1.4.2 (10) one can then apply $R_+^\Omega(\alpha, x)$ to the C^k-function $y \mapsto u(x, y)$. Furthermore, the support of the continuous function $x \mapsto R_+^\Omega(\alpha, x)\, [y \mapsto u(x, y)]$ is contained in $J_+^\Omega(K_1) \cap J_-^\Omega(\mathrm{supp}(y \mapsto u(x, y))) \subset J_+^\Omega(K_1) \cap J_-^\Omega(J_-^\Omega(K_2)) = J_+^\Omega(K_1) \cap J_+^\Omega(K_2)$, which is relatively compact in $\overline{\Omega}$, again by Lemma A.5.3. Hence the function $x \mapsto R_+^\Omega(\alpha, x)\, [y \mapsto u(x, y)]$ has compact support in $\overline{\Omega}$, so that the left-hand side makes sense. Analogously the right-hand side is well defined. Our considerations also show that the integrals depend only on the values of u on $\big(J_+^\Omega(K_1) \cap J_-^\Omega(K_2)\big) \times \big(J_+^\Omega(K_1) \cap J_-^\Omega(K_2)\big)$ which is a relatively compact set. Applying a cut-off function argument we may assume without loss of generality that u has compact support. Proposition 1.4.2 (13) says that the integrals depend holomorphically on α on the domain $\{\Re\mathrm{e}(\alpha) > 0\}$. Therefore it suffices to show the equality for α with sufficiently large real part, which can be done exactly as in the proof of Lemma 1.4.3. $\square$

1.5 Normally hyperbolic operators

Let M be a Lorentzian manifold and let $E \to M$ be a real or complex vector bundle. For a summary on basics concerning linear differential operators see Appendix A.4.

A linear differential operator $P: C^\infty(M,E) \to C^\infty(M,E)$ of second order will be called *normally hyperbolic* if its principal symbol is given by the metric,

$$\sigma_P(\xi) = -\langle \xi, \xi \rangle \cdot \mathrm{id}_{E_x}$$

for all $x \in M$ and all $\xi \in T_x^*M$. In other words, if we choose local coordinates $x^1, \dots, x^n$ on M and a local trivialization of E, then

$$P = -\sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{n} A_j(x) \frac{\partial}{\partial x^j} + B_1(x)$$

where A_j and B_1 are matrix-valued coefficients depending smoothly on x and $(g^{ij})_{ij}$ is the inverse matrix of $(g_{ij})_{ij}$ with $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$.

Example 1.5.1. Let E be the trivial line bundle so that sections in E are just functions. The d'Alembert operator $P = \Box$ is normally hyperbolic because

$$\sigma_{\mathrm{grad}}(\xi) f = f \xi^\sharp, \quad \sigma_{\mathrm{div}}(\xi) X = \xi(X)$$

and so

$$\sigma_\Box(\xi) f = -\sigma_{\mathrm{div}}(\xi) \circ \sigma_{\mathrm{grad}}(\xi) f = -\xi(f\xi^\sharp) = -\langle \xi, \xi \rangle f.$$

Recall that $\xi \mapsto \xi^\sharp$ denotes the isomorphism $T_x^*M \to T_xM$ induced by the Lorentzian metric, compare (1.11).

Example 1.5.2. Let E be a vector bundle and let ∇ be a connection on E. This connection together with the Levi-Civita connection on T^*M induces a connection on $T^*M \otimes E$, again denoted ∇. We define the *connection-d'Alembert operator* $\Box^\nabla$ to be minus the composition of the following three maps

$$C^\infty(M,E) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes E)$$
$$\xrightarrow{\nabla} C^\infty(M, T^*M \otimes T^*M \otimes E) \xrightarrow{\mathrm{tr} \otimes \mathrm{id}_E} C^\infty(M,E)$$

where $\mathrm{tr}: T^*M \otimes T^*M \to \mathbb{R}$ denotes the metric trace, $\mathrm{tr}(\xi \otimes \eta) = \langle \xi, \eta \rangle$. We compute the principal symbol,

$$\sigma_{\Box^\nabla}(\xi)\varphi = -(\mathrm{tr} \otimes \mathrm{id}_E) \circ \sigma_\nabla(\xi) \circ \sigma_\nabla(\xi)(\varphi) = -(\mathrm{tr} \otimes \mathrm{id}_E)(\xi \otimes \xi \otimes \varphi) = -\langle \xi, \xi \rangle \varphi.$$

Hence $\Box^\nabla$ is normally hyperbolic.

Example 1.5.3. Let $E = \Lambda^k T^*M$ be the bundle of k-forms. Exterior differentiation $d: C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ increases the degree by one while the codifferential $\delta: C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k-1} T^*M)$ decreases the degree by one, see [Besse1987, p. 34] for details. While d is independent of the metric, the codifferential δ does depend on the Lorentzian metric. The operator $P = d\delta + \delta d$ is normally hyperbolic.

Example 1.5.4. If M carries a Lorentzian metric and a spin structure, then one can define the spinor bundle ΣM and the Dirac operator

$$D: C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M),$$

see [Bär–Gauduchon–Moroianu2005] or [Baum1981] for the definitions. The principal symbol of D is given by Clifford multiplication,

$$\sigma_D(\xi)\psi = \xi^\sharp \cdot \psi.$$

Hence

$$\sigma_{D^2}(\xi)\psi = \sigma_D(\xi)\sigma_D(\xi)\psi = \xi^\sharp \cdot \xi^\sharp \cdot \psi = -\langle \xi, \xi \rangle \, \psi.$$

Thus $P = D^2$ is normally hyperbolic.

The following lemma is well-known, see e.g. [Baum–Kath1996, Prop. 3.1]. It says that each normally hyperbolic operator is a connection-d'Alembert operator up to a term of order zero.

Lemma 1.5.5. *Let $P : C^\infty(M, E) \to C^\infty(M, E)$ be a normally hyperbolic operator on a Lorentzian manifold M. Then there exists a unique connection ∇ on E and a unique endomorphism field $B \in C^\infty(M, \mathrm{Hom}(E, E))$ such that*

$$P = \Box^\nabla + B.$$

Proof. First we prove uniqueness of such a connection. Let ∇' be an arbitrary connection on E. For any section $s \in C^\infty(M, E)$ and any function $f \in C^\infty(M)$ we get

$$\Box^{\nabla'}(f \cdot s) = f \cdot (\Box^{\nabla'} s) - 2\nabla'_{\mathrm{grad}\, f} s + (\Box f) \cdot s. \tag{1.19}$$

Now suppose that ∇ satisfies the condition in Lemma 1.5.5. Then $B = P - \Box^\nabla$ is an endomorphism field and we obtain

$$f \cdot \big(P(s) - \Box^\nabla s\big) = P(f \cdot s) - \Box^\nabla(f \cdot s).$$

By (1.19) this yields

$$\nabla_{\mathrm{grad}\, f} s = \tfrac{1}{2}\left\{ f \cdot P(s) - P(f \cdot s) + (\Box f) \cdot s \right\}. \tag{1.20}$$

At a given point $x \in M$ every tangent vector $X \in T_x M$ can be written in the form $X = \mathrm{grad}_x f$ for some suitably chosen function f. Thus (1.20) shows that ∇ is determined by P and $\Box$ (which is determined by the Lorentzian metric).

To show existence one could use (1.20) to define a connection ∇ as in the statement. We follow an alternative path. Let ∇' be some connection on E. Since P and $\Box^{\nabla'}$ are both normally hyperbolic operators acting on sections in E, the difference $P - \Box^{\nabla'}$ is a differential operator of first order and can therefore be written in the form

$$P - \Box^{\nabla'} = A' \circ \nabla' + B',$$

for some $A' \in C^\infty(M, \mathrm{Hom}(T^*M \otimes E, E))$ and $B' \in C^\infty(M, \mathrm{Hom}(E, E))$. Set for every vector field X on M and section s in E

$$\nabla_X s := \nabla'_X s - \frac{1}{2} A'(X^\flat \otimes s).$$

This defines a new connection ∇ on E. Let $e_1, \dots, e_n$ be a local Lorentz orthonormal basis of TM. Write as before $\varepsilon_j = \langle e_j, e_j \rangle = \pm 1$. We may assume that at a given point $p \in M$ we have $\nabla_{e_j} e_j(p) = 0$. Then we compute at p

$$\begin{aligned}
\Box^{\nabla'} s + A' \circ \nabla' s &= \sum_{j=1}^n \varepsilon_j \Big\{ - \nabla'_{e_j} \nabla'_{e_j} s + A'(e_j^\flat \otimes \nabla'_{e_j} s) \Big\} \\
&= \sum_{j=1}^n \varepsilon_j \Big\{ - (\nabla_{e_j} + \frac{1}{2} A'(e_j^\flat \otimes \cdot))(\nabla_{e_j} s + \frac{1}{2} A'(e_j^\flat \otimes s)) \\
&\qquad + A'(e_j^\flat \otimes \nabla_{e_j} s) + \frac{1}{2} A'(e_j^\flat \otimes A'(e_j^\flat \otimes s)) \Big\} \\
&= \sum_{j=1}^n \varepsilon_j \Big\{ - \nabla_{e_j} \nabla_{e_j} s - \frac{1}{2} \nabla_{e_j} (A'(e_j^\flat \otimes s)) \\
&\qquad + \frac{1}{2} A'(e_j^\flat \otimes \nabla_{e_j} s) + \frac{1}{4} A'(e_j^\flat \otimes A'(e_j^\flat \otimes s)) \Big\} \\
&= \Box^\nabla s + \frac{1}{4} \sum_{j=1}^n \varepsilon_j \Big\{ A'(e_j^\flat \otimes A'(e_j^\flat \otimes s)) - 2(\nabla_{e_j} A')(e_j^\flat \otimes s) \Big\},
\end{aligned}$$

where ∇ in $\nabla_{e_j} A'$ stands for the induced connection on $\mathrm{Hom}(T^*M \otimes E, E)$. We observe that $Q(s) := \Box^{\nabla'} s + A' \circ \nabla' s - \Box^\nabla s = \frac{1}{4} \sum_{j=1}^n \varepsilon_j \big\{ A'(e_j^\flat \otimes A'(e_j^\flat \otimes s)) - 2(\nabla_{e_j} A')(e_j^\flat \otimes s) \big\}$ is of order zero. Hence

$$P = \Box^{\nabla'} + A' \circ \nabla' + B' = \Box^\nabla s + Q(s) + B'(s)$$

is the desired expression with $B = Q + B'$. $\Box$

The connection in Lemma 1.5.5 will be called the *P-compatible* connection. We shall henceforth always work with the P-compatible connection. We restate (1.20) as a lemma.

Lemma 1.5.6. *Let $P = \Box^\nabla + B$ be normally hyperbolic. For $f \in C^\infty(M)$ and $s \in C^\infty(M, E)$ one gets*

$$P(f \cdot s) = f \cdot P(s) - 2 \nabla_{\mathrm{grad} f} s + \Box f \cdot s. \qquad \Box$$

2 The local theory

Now we start with our detailed study of wave equations. By a wave equation we mean an equation of the form $Pu = f$ where P is a normally hyperbolic operator acting on sections in a vector bundle. The right-hand side f is given and the section u is to be found. In this chapter we deal with local problems, i.e., we try to find solutions defined on sufficiently small domains. This can be understood as a preparation for the global theory which we postpone to the third chapter. Solving wave equations on all of the Lorentzian manifold is, in general, possible only under the geometric assumption of the manifold being globally hyperbolic.

There are various techniques available in the theory of partial differential equations that can be used to settle the local theory. We follow an approach based on Riesz distributions and Hadamard coefficients as in [Günther1988]. The central task is to construct fundamental solutions. This means that one solves the wave equation where the right-hand side f is a delta-distribution.

The construction consists of three steps. First one writes down a formal series in Riesz distributions with unknown coefficients. The wave equation yields recursive relations for these Hadamard coefficients known as transport equations. Since the transport equations are ordinary differential equations along geodesics they can be solved uniquely. There is no reason why the formal solution constructed in this way should be convergent.

In the second step one makes the series convergent by introducing certain cut-off functions. This is similar to the standard proof showing that each formal power series is the Taylor series of some smooth function. Since there are error terms produced by the cut-off functions the result is convergent but no longer solves the wave equation. We call it an approximate fundamental solution.

Thirdly, we turn the approximate fundamental solution into a true one using certain integral operators. Once the existence of fundamental solutions is established one can find solutions to the wave equation for an arbitrary smooth f with compact support. The support of these solutions is contained in the future or in the past of the support of f.

Finally, we show that the formal fundamental solution constructed in the first step is asymptotic to the true fundamental solution. This implies that the singularity structure of the fundamental solution is completely determined by the Hadamard coefficients which are in turn determined by the geometry of the manifold and the coefficients of the operator.

2.1 The formal fundamental solution

In this chapter the underlying Lorentzian manifold will typically be denoted by Ω. Later, in Chapter 3, when we apply the local results Ω will play the role of a small neighborhood of a given point.

Definition 2.1.1. Let Ω be a time-oriented Lorentzian manifold, let $E \to \Omega$ be a vector bundle and let $P : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$ be normally hyperbolic. Let $x \in \Omega$. A *fundamental solution* of P at x is a distribution $F \in \mathcal{D}'(\Omega, E, E_x^*)$ such that

$$PF = \delta_x.$$

In other words, for all $\varphi \in \mathcal{D}(\Omega, E^*)$ we have

$$F[P^*\varphi] = \varphi(x).$$

If $\mathrm{supp}(F(x)) \subset J_+^\Omega(x)$, then we call F an *advanced fundamental solution*, if $\mathrm{supp}(F(x)) \subset J_-^\Omega(x)$, then we call F a *retarded fundamental solution*.

For flat Minkowski space with $P = \Box$ acting on functions Proposition 1.2.4 (3) and (7) show that the Riesz distributions $R_\pm(2)$ are fundamental solutions at $x = 0$. More precisely, $R_+(2)$ is an advanced fundamental solution because its support is contained in $J_+(0)$ and $R_-(2)$ is a retarded fundamental solution.

On a general time-oriented Lorentzian manifold Ω the situation is more complicated even if $P = \Box$. The reason is the factor $\frac{\Box\Gamma_x - 2n}{2\alpha} + 1$ in Proposition 1.4.2 (5) which cannot be evaluated for $\alpha = 0$ unless $\Box\Gamma_x - 2n$ vanished identically. It will turn out that $R_\pm^\Omega(2, x)$ does not suffice to construct fundamental solutions. We will also need Riesz distributions $R_\pm^\Omega(2 + 2k, x)$ for $k \geq 1$.

Let Ω be geodesically starshaped with respect to some fixed $x \in \Omega$ so that the Riesz distributions $R_\pm^\Omega(\alpha, x) = R_\pm^\Omega(\alpha, x)$ are defined. Let $E \to \Omega$ be a real or complex vector bundle and let P be a normally hyperbolic operator P acting on $C^\infty(\Omega, E)$. In this section we start constructing fundamental solutions. We make the following formal ansatz:

$$\mathcal{R}_\pm(x) := \sum_{k=0}^{\infty} V_x^k \; R_\pm^\Omega(2 + 2k, x)$$

where $V_x^k \in C^\infty(\Omega, E \otimes E_x^*)$ are smooth sections yet to be found. For $\varphi \in \mathcal{D}(\Omega, E^*)$ the function $V_x^k \cdot \varphi$ is an E_x^*-valued testfunction and we have $(V_x^k \cdot R_\pm^\Omega(2+2k, x))[\varphi] = R_\pm^\Omega(2+2k, x)[V_x^k \cdot \varphi] \in E_x^*$. Hence each summand $V_x^k \cdot R_\pm^\Omega(2+2k, x)$ is a distribution in $\mathcal{D}'(\Omega, E, E_x^*)$.

By formal termwise differentiation using Lemma 1.5.6 and Proposition 1.4.2 we translate the condition of $\mathcal{R}_\pm(x)$ being a fundamental solution at x into conditions on the V_x^k. To do this let ∇ be the P-compatible connection on E, that is, $P = \Box^\nabla + B$ where $B \in C^\infty(\Omega, \mathrm{End}(E))$, compare Lemma 1.5.5. We compute

$$\begin{aligned} R_\pm^\Omega(0, x) = \delta_x \;&=\; P\mathcal{R}_\pm(x) = \sum_{k=0}^{\infty} P(V_x^k R_\pm^\Omega(2 + 2k, x)) \\ &= \sum_{k=0}^{\infty} \{V_x^k \cdot \Box R_\pm^\Omega(2 + 2k, x) - 2\,\nabla_{\mathrm{grad}\, R_\pm^\Omega(2+2k,x)} V_x^k + P V_x^k \cdot R_\pm^\Omega(2 + 2k, x)\} \end{aligned}$$

$$= V_x^0 \cdot \Box R_\pm^\Omega(2,x) - 2\,\nabla_{\mathrm{grad}\, R_\pm^\Omega(2,x)} V_x^0$$
$$+ \sum_{k=1}^{\infty} \Big\{ V_x^k \cdot \Big(\frac{\frac{1}{2}\Box\Gamma_x - n}{2k} + 1 \Big) R_\pm^\Omega(2k,x) - \tfrac{2}{4k}\,\nabla_{\mathrm{grad}\,\Gamma_x\, R_\pm^\Omega(2k,x)} V_x^k$$
$$+ P\, V_x^{k-1} \cdot R_\pm^\Omega(2k,x) \Big\}$$
$$= V_x^0 \cdot \Box R_\pm^\Omega(2,x) - 2\,\nabla_{\mathrm{grad}\, R_\pm^\Omega(2,x)} V_x^0$$
$$+ \sum_{k=1}^{\infty} \tfrac{1}{2k} \big\{ \big(\tfrac{1}{2}\Box\Gamma_x - n + 2k\big)\, V_x^k - \nabla_{\mathrm{grad}\,\Gamma_x} V_x^k + 2k\, P\, V_x^{k-1} \big\} R_\pm^\Omega(2k,x).$$

Comparing the coefficients of $R_\pm^\Omega(2k,x)$ we get the conditions

$$2\,\nabla_{\mathrm{grad}\, R_\pm^\Omega(2,x)} V_x^0 - \Box R_\pm^\Omega(2,x) \cdot V_x^0 + R_\pm^\Omega(0,x) = 0 \quad \text{and} \tag{2.1}$$
$$\nabla_{\mathrm{grad}\,\Gamma_x} V_x^k - \big(\tfrac{1}{2}\Box\Gamma_x - n + 2k\big)\, V_x^k = 2k\, P\, V_x^{k-1} \quad \text{for } k \geq 1. \tag{2.2}$$

We take a look at what condition (2.2) would mean for $k = 0$. We multiply this equation by $R_\pm^\Omega(\alpha,x)$:

$$\nabla_{\mathrm{grad}\,\Gamma_x\, R_\pm^\Omega(\alpha,x)} V_x^0 - \big(\tfrac{1}{2}\Box\Gamma_x - n\big)\; V_x^0 \cdot R_\pm^\Omega(\alpha,x) = 0.$$

By Proposition 1.4.2 (4) and (5) we obtain

$$\nabla_{2\alpha\,\mathrm{grad}\, R_\pm^\Omega(\alpha+2,x)} V_x^0 - \big(\alpha \Box R_\pm^\Omega(\alpha+2,x) - \alpha R_\pm^\Omega(\alpha,x)\big)\, V_x^0 = 0.$$

Division by α and the limit $\alpha \to 0$ yield

$$2\nabla_{\mathrm{grad}\, R_\pm^\Omega(2,x)} V_x^0 - \big(\Box R_\pm^\Omega(2,x) - R_\pm^\Omega(0,x)\big)\, V_x^0 = 0.$$

Therefore we recover condition (2.1) if and only if $V_x^0(x) = \mathrm{id}_{E_x}$.

In order to obtain formal fundamental solutions $\mathcal{R}_\pm(x)$ for P we hence need $V_x^k \in C^\infty(\Omega, E \otimes E_x^*)$ satisfying

$$\nabla_{\mathrm{grad}\,\Gamma_x} V_x^k - \big(\tfrac{1}{2}\Box\Gamma_x - n + 2k\big)\, V_x^k = 2k\, P\, V_x^{k-1} \tag{2.3}$$

for all $k \geq 0$ with "initial condition" $V_x^0(x) = \mathrm{id}_{E_x}$. In particular, we have the same conditions on V_x^k for $\mathcal{R}_+(x)$ and for $\mathcal{R}_-(x)$. Equations (2.3) are known as *transport equations*.

2.2 Uniqueness of the Hadamard coefficients

This and the next section are devoted to uniqueness and existence of solutions to the transport equations.

Definition 2.2.1. Let Ω be time-oriented and geodesically starshaped with respect to $x \in \Omega$. Sections $V_x^k \in C^\infty(\Omega, E \otimes E_x^*)$ are called *Hadamard coefficients* for P at x if they satisfy the transport equations (2.3) for all $k \geq 0$ and $V_x^0(x) = \mathrm{id}_{E_x}$. Given Hadamard coefficients V_x^k for P at x we call the formal series

$$\mathcal{R}_+(x) = \sum_{k=0}^{\infty} V_x^k \cdot R_+^\Omega(2+2k, x)$$

a *formal advanced fundamental solution for P at x* and

$$\mathcal{R}_-(x) = \sum_{k=0}^{\infty} V_x^k \cdot R_-^\Omega(2+2k, x)$$

a *formal retarded fundamental solution for P at x*.

In this section we show uniqueness of the Hadamard coefficients (and hence of the formal fundamental solutions $\mathcal{R}_\pm(x)$) by deriving explicit formulas for them. These formulas will also be used in the next section to prove existence.

For $y \in \Omega$ we denote the ∇-parallel translation along the (unique) geodesic from x to y by

$$\Pi_y^x \colon E_x \to E_y.$$

We have $\Pi_x^x = \mathrm{id}_{E_x}$ and $(\Pi_y^x)^{-1} = \Pi_x^y$. Note that the map $\Phi \colon \Omega \times [0,1] \to \Omega$, $\Phi(y,s) = \exp_x(s \cdot \exp_x^{-1}(y))$, is well defined and smooth since Ω is geodesically starshaped with respect to x.

Lemma 2.2.2. *Let V_x^k be Hadamard coefficients for P at x. Then they are given by*

$$V_x^0(y) = \mu_x^{-1/2}(y)\Pi_y^x \tag{2.4}$$

and for $k \geq 1$

$$V_x^k(y) = -k\,\mu_x^{-1/2}(y)\,\Pi_y^x \int_0^1 \mu_x^{1/2}(\Phi(y,s))s^{k-1}\,\Pi_x^{\Phi(y,s)}(P V_x^{k-1}(\Phi(y,s)))\,ds. \tag{2.5}$$

Proof. We put $\rho := \sqrt{|\Gamma_x|}$. On $\Omega \setminus C(x)$ where $C(x) = \exp_x(C(0))$ is the light cone of x we have $\Gamma_x(y) = -\varepsilon\rho^2(y)$ where $\varepsilon = 1$ if $\exp_x^{-1}(y)$ is spacelike and $\varepsilon = -1$ if $\exp_x^{-1}(y)$ is timelike. Using the identities $\frac{1}{2}\Box\Gamma_x - n = -\frac{1}{2}\partial_{\mathrm{grad}\,\Gamma_x}\log\mu_x = -\partial_{\mathrm{grad}\,\Gamma_x}\log(\mu_x^{1/2})$ from Lemma 1.3.19 (3) and $\partial_{\mathrm{grad}\,\Gamma_x}(\log\rho^k) = k\,\partial_{-2\varepsilon\rho\,\mathrm{grad}\,\rho}\log\rho = -2\varepsilon k\rho\frac{\partial_{\mathrm{grad}\,\rho}\rho}{\rho} = -2k$ we reformulate (2.3):

$$\nabla_{\mathrm{grad}\,\Gamma_x} V_x^k + \partial_{\mathrm{grad}\,\Gamma_x}\log\left(\mu_x^{1/2}\cdot\rho^k\right)V_x^k = 2k\,P V_x^{k-1}.$$

This is equivalent to

$$\begin{aligned}\nabla_{\mathrm{grad}\,\Gamma_x}\left(\mu_x^{1/2}\cdot\rho^k\cdot V_x^k\right) &= \mu_x^{1/2}\cdot\rho^k\nabla_{\mathrm{grad}\,\Gamma_x}V_x^k + \partial_{\mathrm{grad}\,\Gamma_x}\left(\mu_x^{1/2}\cdot\rho^k\right)V_x^k \\ &= \mu_x^{1/2}\cdot\rho^k\cdot 2k\cdot P V_x^{k-1}.\end{aligned} \tag{2.6}$$

For $k = 0$ one has $\nabla_{\operatorname{grad}\Gamma_x}(\mu_x^{1/2}V_x^0) = 0$. Hence $\mu_x^{1/2}V_x^0$ is ∇-parallel along the timelike and spacelike geodesics starting in x. By continuity it is ∇-parallel along any geodesic starting at x. Since $\mu_x^{1/2}(x)V_x^0(x) = 1 \cdot \mathrm{id}_{E_x} = \Pi_x^x$ we conclude $\mu_x^{1/2}(y)V_x^0(y) = \Pi_y^x$ for all $y \in \Omega$. This shows (2.4).

Next we determine V_x^k for $k \geq 1$. We consider some point $y \in \Omega \setminus C(x)$ outside the light cone of x. We put $\eta := \exp_x^{-1}(y)$. Then $c(t) := \exp_x(e^{2t} \cdot \eta)$ gives a reparametrization of the geodesic $\beta(t) = \exp_x(t\eta)$ from x to y such that $\dot{c}(t) = 2e^{2t}\dot{\beta}(e^{2t})$. By Lemma 1.3.19 (1)

$$\begin{aligned}\langle \dot{c}(t), \dot{c}(t)\rangle &= 4e^{4t}\langle \dot{\beta}(e^{2t}), \dot{\beta}(e^{2t})\rangle \\ &= 4e^{4t}\langle \eta, \eta\rangle = -4\gamma(e^{2t}\eta) \\ &= -4\Gamma_x(c(t)) = \langle \operatorname{grad}\Gamma_x, \operatorname{grad}\Gamma_x\rangle.\end{aligned}$$

Thus c is an integral curve of the vector field $-\operatorname{grad}\Gamma_x$. Equation (2.6) can be rewritten as

$$-\frac{\nabla}{dt}\big(\mu_x^{1/2} \cdot \rho^k \cdot V_x^k\big)(c(t)) = \big(\mu_x^{1/2} \cdot \rho^k \cdot 2k \cdot P V_x^{k-1}\big)(c(t)),$$

which we can solve explicitly:

$$\begin{aligned}&\big(\mu_x^{1/2} \cdot \rho^k \cdot V_x^k\big)(c(t)) \\ &\quad = -\Pi_{c(t)}^x\left(\int_{-\infty}^t \Pi_x^{c(\tau)}\big(\mu_x^{1/2} \cdot \rho^k \cdot 2k \cdot P V_x^{k-1}\big)(c(\tau))d\tau\right) \\ &\quad = -2k\,\Pi_{c(t)}^x\left(\int_{-\infty}^t \mu_x^{1/2}(c(\tau))\rho(c(\tau))^k\,\Pi_x^{c(\tau)}\big(P V_x^{k-1}(c(\tau))\big)d\tau\right).\end{aligned}$$

We have

$$\rho(c(\tau))^k = \rho(\exp_x(e^{2\tau}\eta))^k = |\gamma(e^{2\tau}\eta)|^{k/2} = |e^{4\tau}\gamma(\eta)|^{k/2} = e^{2k\tau}|\gamma(\eta)|^{k/2}.$$

Since $y \notin C(x)$ we can divide by $|\gamma(\eta)| \neq 0$:

$$\begin{aligned}&e^{2kt}\big(\mu_x^{1/2}V_x^k\big)(c(t)) \\ &\quad = -2k\,\Pi_{c(t)}^x\left(\int_{-\infty}^t \mu_x^{1/2}(c(\tau))\,e^{2k\tau}\,\Pi_x^{c(\tau)}\big(P V_x^{k-1}(c(\tau))\big)\,d\tau\right) \\ &\quad = -2k\,\Pi_{c(t)}^x\int_0^{e^{2t}} \mu_x^{1/2}(\exp_x(s\cdot\eta))\,s^k\,\Pi_x^{\exp_x(s\cdot\eta)}(P V_x^{k-1}(\exp_x(s\cdot\eta)))\,\frac{ds}{2s}\end{aligned}$$

where we used the substitution $s = e^{2\tau}$. For $t = 0$ this yields (2.5). □

Corollary 2.2.3. *Let Ω be time-oriented and geodesically starshaped with respect to $x \in \Omega$. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over Ω.*

Then the Hadamard coefficients V_x^k for P at x are unique for all $k \geq 0$. □

2.3 Existence of the Hadamard coefficients

Let Ω be time-oriented and geodesically starshaped with respect to $x \in \Omega$. Let P be a normally hyperbolic operator acting on sections in a real or complex vector bundle E over Ω. To construct Hadamard coefficients for P at x we use formulas (2.4) and (2.5) obtained in the previous section as definitions:

$$V_x^0(y) := \mu_x^{-1/2}(y) \cdot \Pi_y^x$$

and

$$V_x^k(y) := -k\, \mu_x^{-1/2}(y)\, \Pi_y^x \int_0^1 \mu_x^{1/2}(\Phi(y,s)) s^{k-1}\, \Pi_x^{\Phi(y,s)} \big(P V_x^{k-1}(\Phi(y,s))\big)\, ds.$$

We observe that this defines smooth sections $V_x^k \in C^\infty(\Omega, E \otimes E_x^*)$. We have to check for all $k \geq 0$

$$\nabla_{\mathrm{grad}\,\Gamma_x} \big(\mu_x^{1/2}\, \rho^k\, V_x^k\big) = \mu_x^{1/2}\, \rho^k \cdot 2k \cdot P V_x^{k-1}, \tag{2.7}$$

from which Equation (2.3) follows as we have already seen. For $k = 0$ Equation (2.7) obviously holds:

$$\nabla_{\mathrm{grad}\,\Gamma_x} \big(\mu_x^{1/2}\, V_x^0\big) = \nabla_{\mathrm{grad}\,\Gamma_x} \Pi = 0.$$

For $k \geq 1$ we check:

$$\begin{aligned}
&\nabla_{\mathrm{grad}\,\Gamma_x} \big(\mu_x^{1/2}\, \rho^k\, V_x^k\big)(y) \\
&\quad = -k \nabla_{\mathrm{grad}\,\Gamma_x} \Pi_y^x \int_0^1 \mu_x^{1/2}(\Phi(y,s))\, \underbrace{\rho(y)^k \cdot s^k}_{=\rho(\Phi(y,s))^k}\, \Pi_x^{\Phi(y,s)} P V_x^{k-1}(\Phi(y,s))\, \frac{ds}{s} \\
&\quad = -k \nabla_{\mathrm{grad}\,\Gamma_x} \Pi_y^x \int_0^1 \big(\mu_x^{1/2} \rho^k\, \Pi^x_{\Phi(y,s)} (P V_x^{k-1})\big)(\Phi(y,s))\, \frac{ds}{s} \\
&\overset{s=e^{2\tau}}{=} -2k \nabla_{\mathrm{grad}\,\Gamma_x} \Pi_y^x \int_{-\infty}^0 \big(\mu_x^{1/2} \rho^k\, \Pi^x_{\Phi(y,s)} (P V_x^{k-1})\big) \underbrace{(\Phi(y, e^{2\tau}))}_{\substack{\text{integral curve} \\ \text{for } -\mathrm{grad}\,\Gamma_x}}\, d\tau \\
&\quad = 2k\ \Pi_y^x\ \frac{d}{dt}\Big|_{t=0} \int_{-\infty}^0 \big(\mu_x^{1/2} \rho^k\, \Pi^x_{\Phi(y,s)} (P V_x^{k-1})\big)\, \big(\Phi\, (\Phi(y, e^{2t}), e^{2\tau})\big)\ d\tau \\
&\quad = 2k\ \Pi_y^x\ \frac{d}{dt}\Big|_{t=0} \int_{-\infty}^0 \big(\mu_x^{1/2} \rho^k\, \Pi^x_{\Phi(y,s)} (P V_x^{k-1})\big)\big(\Phi\big(y, e^{2(\tau+t)}\big)\big)\, d\tau \\
&\overset{\tau'=\tau+t}{=} 2k\ \Pi_y^x\ \frac{d}{dt}\Big|_{t=0} \int_{-\infty}^t \big(\mu_x^{1/2} \rho^k\, \Pi^x_{\Phi(y,s)} (P V_x^{k-1})\big)\big(\Phi\big(y, e^{2\tau'}\big)\big)\, d\tau' \\
&\quad = 2k\ \Pi_y^x\ \big(\mu_x^{1/2} \rho^k\, \Pi^x_{\Phi(y,s)} (P V_x^{k-1})\big)\underbrace{(\Phi(y, e^0))}_{=y} \\
&\quad = 2k\, \mu_x^{1/2}(y)\, \rho^k(y)\, \big(P V_x^{k-1}\big)(y)
\end{aligned}$$

which is (2.7). This shows the existence of the Hadamard coefficients and, therefore, we have found formal fundamental solutions $\mathcal{R}_\pm(x)$ for P at fixed $x \in \Omega$.

Now we let x vary. We assume there exists an open subset $U \subset \Omega$ such that Ω is geodesically starshaped with respect to all $x \in U$. This ensures that the Riesz distributions $R_\pm^\Omega(\alpha, x)$ are defined for all $x \in U$. We write $V_k(x,y) := V_x^k(y)$ for the Hadamard coefficients at x. Thus $V_k(x,y) \in \mathrm{Hom}(E_x, E_y) = E_x^* \otimes E_y$. The explicit formulas (2.4) and (2.5) show that the Hadamard coefficients V_k also depend smoothly on x, i.e.,

$$V_k \in C^\infty(U \times \Omega, E^* \boxtimes E).$$

Recall that $E^* \boxtimes E$ is the bundle with fiber $(E^* \boxtimes E)_{(x,y)} = E_x^* \otimes E_y$. We have formal fundamental solutions for P at all $x \in U$:

$$\mathcal{R}_\pm(x) = \sum_{k=0}^{\infty} V_k(x,\cdot)\, R_\pm^\Omega(2+2k, x).$$

We summarize our results about Hadamard coefficients obtained so far.

Proposition 2.3.1. *Let Ω be a Lorentzian manifold, let $U \subset \Omega$ be a nonempty open subset such that Ω is geodesically starshaped with respect to all points $x \in U$. Let $P = \Box^\nabla + B$ be a normally hyperbolic operator acting on sections in a real or complex vector bundle over Ω. Denote the ∇-parallel transport by Π.*

Then at each $x \in U$ there are unique Hadamard coefficients $V_k(x,\cdot)$ for P, $k \geq 0$. They are smooth, $V_k \in C^\infty(U \times \Omega, E^* \boxtimes E)$, and are given by

$$V_0(x,y) = \mu_x^{-1/2}(y) \cdot \Pi_y^x$$

and for $k \geq 1$

$$V_k(x,y) = -k\,\mu_x^{-1/2}(y)\,\Pi_y^x \int_0^1 \mu_x^{1/2}(\Phi(y,s)) s^{k-1}\,\Pi_x^{\Phi(y,s)} (P_{(2)} V_{k-1})(x, \Phi(y,s))\,ds$$

where $P_{(2)}$ denotes the action of P on the second variable of V_{k-1}. □

These formulas become particularly simple along the diagonal, i.e., for $x = y$. We have for any normally hyperbolic operator P

$$V_0(x,x) = \mu_x(x)^{-1/2}\Pi_x^x = \mathrm{id}_{E_x}.$$

For $k \geq 1$ we get

$$\begin{aligned} V_k(x,x) &= -k \underbrace{\mu_x^{-1/2}(x)}_{=1} \cdot 1 \cdot \underbrace{\Pi_x^x}_{=\mathrm{id}} \int_0^1 s^{k-1} \underbrace{\Pi_x^x}_{=\mathrm{id}} (P_{(2)} V_{k-1})(x,x) \mu_x^{-1/2}(x) ds \\ &= -(P_{(2)} V_{k-1})(x,x). \end{aligned}$$

We compute $V_1(x,x)$ for $P = \Box^\nabla + B$. By (2.5) and Lemma 1.5.6 we have

$$\begin{aligned} V_1(x,x) &= -(P_{(2)}V_0)(x,x) \\ &= -P(\mu_x^{-1/2}\Pi^x_\bullet)(x) \\ &= -\mu_x^{-1/2}(x)\cdot P(\Pi^x_\bullet)(x) + 2\nabla_{\underbrace{\mathrm{grad}\,\mu_x(x)}_{=0}}\Pi^x_\bullet(x) - (\Box\mu_x^{-1/2})(x)\cdot \mathrm{id}_{E_x} \\ &= -(\Box^\nabla + B)(\Pi^x_\bullet)(x) - (\Box\mu_x^{-1/2})(x)\cdot \mathrm{id}_{E_x} \\ &= -B|_x - (\Box\mu_x^{-1/2})(x)\cdot \mathrm{id}_{E_x}. \end{aligned}$$

From Corollary 1.3.18 we conclude

$$V_1(x,x) = \frac{\mathrm{scal}(x)}{6}\mathrm{id}_{E_x} - B|_x.$$

Remark 2.3.2. We compare our definition of Hadamard coefficients with the definition used in [Günther1988] and in [Baum–Kath1996]. In [Günther1988, Chap. 3, Prop. 1.3] Hadamard coefficients U_k are solutions of the differential equations

$$\mathrm{L}[\Gamma_x, U_k(x,\cdot)] + (M(x,\cdot) + 2k)\,U_k(x,\cdot) = -P\,U_{k-1}(x,\cdot) \tag{2.8}$$

with initial conditions $U_0(x) = \mathrm{id}_{E_x}$, where, in our terminology, $\mathrm{L}[f,\cdot]$ denotes $-\nabla_{\mathrm{grad}\,f}(\cdot)$ for the P-compatible connection ∇, and $M(x,\cdot) = \frac{1}{2}\Box\Gamma_x - n$. Hence (2.8) reads as

$$-\nabla_{\mathrm{grad}\,\Gamma_x}U_k(x,\cdot) + \big(\tfrac{1}{2}\Box\Gamma_x - n + 2k\big)\,U_k(x,\cdot) = -P\,U_{k-1}(x,\cdot).$$

We recover our defining equations (2.2) after the substitution

$$U_k = \tfrac{1}{2^k\cdot k!}V_k.$$

2.4 True fundamental solutions on small domains

In this section we show existence of "true" fundamental solutions in the sense of Definition 2.1.1 on sufficiently small causal domains in a time-oriented Lorentzian manifold M. Assume that $\Omega' \subset M$ is a geodesically convex open subset. We then have the Hadamard coefficients $V_j \in C^\infty(\Omega'\times\Omega', E^*\boxtimes E)$ and for all $x \in \Omega'$ the formal fundamental solutions

$$\mathcal{R}_\pm(x) = \sum_{j=0}^{\infty} V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x).$$

Fix an integer $N \geq \frac{n}{2}$ where n is the dimension of the manifold M. Then for all $j \geq N$ the distribution $R_\pm^{\Omega'}(2+2j,x)$ is a continuous function on Ω'. Hence we can split the formal fundamental solutions

$$\mathcal{R}_\pm(x) = \sum_{j=0}^{N-1} V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x) + \sum_{j=N}^{\infty} V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x)$$

where $\sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)$ is a well-defined E_x^*-valued distribution in E over Ω' and $\sum_{j=N}^{\infty} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)$ is a formal sum of continuous sections, $V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x) \in C^0(\Omega', E_x^* \otimes E)$ for $j \geq N$.

Using suitable cut-offs we will now replace the infinite formal part of the series by a convergent series. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a smooth function vanishing outside $[-1,1]$, such that $\sigma \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $0 \leq \sigma \leq 1$ everywhere. We need the following elementary lemma.

Lemma 2.4.1. *For every $l \in \mathbb{N}$ and every $\beta \geq l+1$ there exists a constant $c(l,\beta)$ such that for all $0 < \varepsilon \leq 1$ we have*

$$\left\| \frac{d^l}{dt^l}(\sigma(t/\varepsilon)t^\beta) \right\|_{C^0(\mathbb{R})} \leq \varepsilon \cdot c(l,\beta) \cdot \|\sigma\|_{C^l(\mathbb{R})}.$$

Proof.

$$\begin{aligned}
&\left\| \frac{d^l}{dt^l}(\sigma(t/\varepsilon)t^\beta) \right\|_{C^0(\mathbb{R})} \\
&\quad \leq \sum_{m=0}^{l} \binom{l}{m} \left\| \frac{1}{\varepsilon^m}\sigma^{(m)}(t/\varepsilon)\cdot \beta(\beta-1)\dots(\beta-l+m+1)t^{\beta-l+m} \right\|_{C^0(\mathbb{R})} \\
&\quad = \sum_{m=0}^{l} \binom{l}{m} \cdot \beta(\beta-1)\dots(\beta-l+m+1)\varepsilon^{\beta-l} \left\| (t/\varepsilon)^{\beta-l+m}\sigma^{(m)}(t/\varepsilon) \right\|_{C^0(\mathbb{R})}.
\end{aligned}$$

Now $\sigma^{(m)}(t/\varepsilon)$ vanishes for $|t|/\varepsilon \geq 1$ and thus $\|(t/\varepsilon)^{\beta-m+l}\sigma^{(m)}(t/\varepsilon)\|_{C^0(\mathbb{R})} \leq \|\sigma^{(m)}\|_{C^0(\mathbb{R})}$. Moreover, $\beta - l \geq 1$, hence $\varepsilon^{\beta-l} \leq \varepsilon$. Therefore

$$\begin{aligned}
\left\| \frac{d^l}{dt^l}(\sigma(t/\varepsilon)t^\beta) \right\|_{C^0(\mathbb{R})} &\leq \varepsilon \sum_{m=0}^{l} \binom{l}{m} \cdot \beta(\beta-1)\dots(\beta-l+m+1) \|\sigma^{(m)}\|_{C^0(\mathbb{R})} \\
&\leq \varepsilon\, c(l,\beta)\, \|\sigma\|_{C^l(\mathbb{R})}.
\end{aligned}$$

□

We define $\Gamma \in C^\infty(\Omega' \times \Omega', \mathbb{R})$ by $\Gamma(x,y) := \Gamma_x(y)$ where Γ_x is as in (1.17). Note that $\Gamma(x,y) = 0$ if and only if the geodesic joining x and y in Ω' is lightlike. In other words, $\Gamma^{-1}(0) = \bigcup_{x\in\Omega'}(C_+^{\Omega'}(x) \cup C_-^{\Omega'}(x))$.

Lemma 2.4.2. *Let $\Omega \subset\subset \Omega'$ be a relatively compact open subset. Then there exists a sequence of $\varepsilon_j \in (0,1]$, $j \geq N$, such that for each $k \geq 0$ the series*

$$\begin{aligned}
(x,y) \mapsto &\sum_{j=N+k}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y) \\
&= \begin{cases} \sum_{j=N+k}^{\infty} C(2+2j,n)\sigma(\Gamma(x,y)/\varepsilon_j)V_j(x,y)\,\Gamma(x,y)^{j+1-n/2}, & \textit{if } y \in J_\pm^{\Omega'}(x) \\ 0, & \textit{otherwise} \end{cases}
\end{aligned}$$

converges in $C^k(\bar\Omega \times \bar\Omega, E^* \boxtimes E)$*. In particular, the series*

$$(x,y) \mapsto \sum_{j=N}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y)$$

defines a continuous section over $\bar\Omega \times \bar\Omega$ *and a smooth section over* $(\bar\Omega \times \bar\Omega) \setminus \Gamma^{-1}(0)$.

Proof. For $j \geq N \geq \frac{n}{2}$ the exponent in $\Gamma(x,y)^{j+1-n/2}$ is positive. Therefore the piecewise definition of the j-th summand yields a continuous section over Ω'.

The factor $\sigma(\Gamma(x,y)/\varepsilon_j)$ vanishes whenever $\Gamma(x,y) \geq \varepsilon_j$. Hence for $j \geq N \geq \frac{n}{2}$ and $0 < \varepsilon_j \leq 1$

$$\begin{aligned}
&\left\| (x,y) \mapsto \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y) \right\|_{C^0(\bar\Omega\times\bar\Omega)} \\
&\quad \leq C(2+2j,n)\ \|V_j\|_{C^0(\bar\Omega\times\bar\Omega)}\ \varepsilon_j^{j+1-n/2} \\
&\quad \leq C(2+2j,n)\ \|V_j\|_{C^0(\bar\Omega\times\bar\Omega)}\ \varepsilon_j .
\end{aligned}$$

Hence if we choose $\varepsilon_j \in (0,1]$ such that

$$C(2+2j,n)\ \|V_j\|_{C^0(\bar\Omega\times\bar\Omega)}\ \varepsilon_j < 2^{-j},$$

then the series converges in the C^0-norm and therefore defines a continuous section.

For $k \geq 0$ and $j \geq N + k \geq \frac{n}{2} + k$ the function $\Gamma^{j+1-\frac{n}{2}}$ vanishes to $(k+1)$-st order along $\Gamma^{-1}(0)$. Thus the j-th summand in the series is of regularity C^k. Writing $\sigma_j(t) := \sigma(t/\varepsilon_j) t^{j+1-n/2}$ we know from Lemma 2.4.1 that

$$\|\sigma_j\|_{C^k(\mathbb{R})} \leq \varepsilon_j \cdot c_1(k,j,n) \cdot \|\sigma\|_{C^k(\mathbb{R})}$$

where here and henceforth $c_1, c_2, \dots$ denote certain universal positive constants whose precise values are of no importance. Using Lemmas 1.1.11 and 1.1.12 we obtain

$$\begin{aligned}
&\left\| (x,y) \mapsto \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y) \right\|_{C^k(\bar\Omega\times\bar\Omega)} \\
&\quad \leq C(2+2j,n) \|(\sigma_j \circ \Gamma) \cdot V_j\|_{C^k(\bar\Omega\times\bar\Omega)} \\
&\quad \leq c_2(k,j,n) \cdot \|\sigma_j \circ \Gamma\|_{C^k(\bar\Omega\times\bar\Omega)} \cdot \|V_j\|_{C^k(\bar\Omega\times\bar\Omega)} \\
&\quad \leq c_3(k,j,n) \cdot \|\sigma_j\|_{C^k(\mathbb{R})} \cdot \max_{\ell=0,\dots,k} \|\Gamma\|^{\ell}_{C^k(\bar\Omega\times\bar\Omega)} \cdot \|V_j\|_{C^k(\bar\Omega\times\bar\Omega)} \\
&\quad \leq c_4(k,j,n) \cdot \varepsilon_j \cdot \|\sigma\|_{C^k(\mathbb{R})} \cdot \max_{\ell=0,\dots,k} \|\Gamma\|^{\ell}_{C^k(\bar\Omega\times\bar\Omega)} \cdot \|V_j\|_{C^k(\bar\Omega\times\bar\Omega)}.
\end{aligned}$$

Hence if we add the (finitely many) conditions on ε_j that

$$c_4(k,j,n) \cdot \varepsilon_j \cdot \|V_j\|_{C^k(\bar\Omega\times\bar\Omega)} \leq 2^{-j}$$

for all $k \leq j - N$, then we have for fixed k

$$\begin{aligned}
&\left\| (x,y) \mapsto \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y) \right\|_{C^k(\bar\Omega\times\bar\Omega)} \\
&\quad \leq 2^{-j} \cdot \|\sigma\|_{C^k(\mathbb{R})} \cdot \max_{\ell=0,\dots,k} \|\Gamma\|^{\ell}_{C^k(\bar\Omega\times\bar\Omega)}
\end{aligned}$$

for all $j \geq N+k$. Thus the series

$$(x,y) \mapsto \sum_{j=N+k}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y)$$

converges in $C^k(\bar{\Omega} \times \bar{\Omega}, E^* \boxtimes E)$. All summands $\sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y)$ are smooth on $\bar{\Omega} \times \bar{\Omega} \setminus \Gamma^{-1}(0)$, thus

$$\begin{aligned}(x,y) \mapsto \sum_{j=N}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y) \\ = \sum_{j=N}^{N+k-1} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y) \\ + \sum_{j=N+k}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y)\end{aligned}$$

is C^k for all k, hence smooth on $(\bar{\Omega} \times \bar{\Omega}) \setminus \Gamma^{-1}(0)$. □

Define distributions $\tilde{\mathcal{R}}_+(x)$ and $\tilde{\mathcal{R}}_-(x)$ by

$$\tilde{\mathcal{R}}_\pm(x) := \sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x) + \sum_{j=N}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x).$$

By Lemma 2.4.2 and the properties of Riesz distributions we know that

$$\operatorname{supp}(\tilde{\mathcal{R}}_\pm(x)) \subset J_\pm^{\Omega'}(x), \tag{2.9}$$

$$\operatorname{sing\,supp}(\tilde{\mathcal{R}}_\pm(x)) \subset C_\pm^{\Omega'}(x), \tag{2.10}$$

and that $\operatorname{ord}(\tilde{\mathcal{R}}_\pm(x)) \leq n+1$.

Lemma 2.4.3. *The ε_j in Lemma 2.4.2 can be chosen such that in addition to the assertion in Lemma 2.4.2 we have on Ω*

$$P_{(2)} \tilde{\mathcal{R}}_\pm(x) = \delta_x + K_\pm(x,\cdot) \tag{2.11}$$

with smooth $K_\pm \in C^\infty(\bar{\Omega} \times \bar{\Omega}, E^* \boxtimes E)$.

Proof. From properties (2.1) and (2.2) of the Hadamard coefficients we know

$$P_{(2)}\Big(\sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)\Big) = \delta_x + (P_{(2)} V_{N-1}(x,\cdot)) R_\pm^{\Omega'}(2N,x). \tag{2.12}$$

Moreover, by Lemma 1.1.10 we may interchange P with the infinite sum and we get

$$
\begin{aligned}
P_{(2)}\Big(\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\,R_{\pm}^{\Omega'}(2+2j,x)\Big)
&= \sum_{j=N}^{\infty} P_{(2)}\big(\sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\,R_{\pm}^{\Omega'}(2+2j,x)\big)\\
&= \sum_{j=N}^{\infty}\big(\Box_{(2)}(\sigma(\Gamma(x,\cdot)/\varepsilon_j))V_j(x,\cdot)R_{\pm}^{\Omega'}(2+2j,x)\\
&\qquad - 2\nabla^{(2)}_{\mathrm{grad}_{(2)}\sigma(\Gamma(x,\cdot)/\varepsilon_j)}(V_j(x,\cdot)R_{\pm}^{\Omega'}(2+2j,x))\\
&\qquad + \sigma(\Gamma(x,\cdot)/\varepsilon_j)P_{(2)}(V_j(x,\cdot)R_{\pm}^{\Omega'}(2+2j,x))\big).
\end{aligned}
$$

Here and in the following $\Box_{(2)}$, $\mathrm{grad}_{(2)}$, and $\nabla^{(2)}$ indicate that the operators are applied with respect to the y-variable just as for $P_{(2)}$.

Abbreviating $\Sigma_1 := \sum_{j=N}^{\infty}\Box_{(2)}(\sigma(\Gamma(x,\cdot)/\varepsilon_j))V_j(x,\cdot)R_{\pm}^{\Omega'}(2+2j,x)$ and $\Sigma_2 := -2\sum_{j=N}^{\infty}\nabla^{(2)}_{\mathrm{grad}_{(2)}\sigma(\Gamma(x,\cdot)/\varepsilon_j)}(V_j(x,\cdot)R_{\pm}^{\Omega'}(2+2j,x))$ we have

$$
\begin{aligned}
P_{(2)}\Big(\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\,R_{\pm}^{\Omega'}(2+2j,x)\Big)
&= \Sigma_1+\Sigma_2+\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)P_{(2)}(V_j(x,\cdot)R_{\pm}^{\Omega'}(2+2j,x))\\
&= \Sigma_1+\Sigma_2+\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)\big((P_{(2)}V_j(x,\cdot))R_{\pm}^{\Omega'}(2+2j,x)\\
&\qquad - 2\nabla^{(2)}_{\mathrm{grad}_{(2)}R_{\pm}^{\Omega'}(2+2j,x)}V_j(x,\cdot) + V_j(x,\cdot)\Box_{(2)}R_{\pm}^{\Omega'}(2+2j,x)\big).
\end{aligned}
$$

Properties (2.1) and (2.2) of the Hadamard coefficients tell us

$$
\begin{aligned}
V_j(x,\cdot)\Box_{(2)}R_{\pm}^{\Omega'}(2+2j,x) - 2\nabla^{(2)}_{\mathrm{grad}_{(2)}R_{\pm}^{\Omega'}(2+2j,x)}V_j(x,\cdot)\\
= -P_{(2)}(V_{j-1}(x,\cdot)R_{\pm}^{\Omega'}(2+2j,x))
\end{aligned}
$$

and hence

$$
\begin{aligned}
P_{(2)}\Big(\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\,R_{\pm}^{\Omega'}(2+2j,x)\Big)
&= \Sigma_1+\Sigma_2+\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)\Big((P_{(2)}V_j(x,\cdot))R_{\pm}^{\Omega'}(2+2j,x)\\
&\qquad - P_{(2)}V_{j-1}R_{\pm}^{\Omega'}(2j,x)\Big)
\end{aligned}
$$

$$= \Sigma_1 + \Sigma_2 - \sigma(\Gamma(x,\cdot)/\varepsilon_N) P_{(2)} V_{N-1} R_\pm^{\Omega'}(2N,x) + \sum_{j=N}^{\infty} \big(\sigma(\Gamma(x,\cdot)/\varepsilon_j) - \sigma(\Gamma(x,\cdot)/\varepsilon_{j+1})\big) \left(P_{(2)} V_j(x,\cdot)\right) R_\pm^{\Omega'}(2+2j,x).$$

Putting

$$\Sigma_3 := \sum_{j=N}^{\infty} \big(\sigma(\Gamma(x,\cdot)/\varepsilon_j) - \sigma(\Gamma(x,\cdot)/\varepsilon_{j+1})\big) \left(P_{(2)} V_j(x,\cdot)\right) R_\pm^{\Omega'}(2+2j,x)$$

and combining with (2.12) yields

$$P_{(2)} \widetilde{\mathcal{R}}_\pm(x) - \delta_x = (1-\sigma(\Gamma(x,\cdot)/\varepsilon_{N-1})) P_{(2)} V_{N-1}(x,\cdot) R_\pm^{\Omega'}(2N,x) + \Sigma_1 + \Sigma_2 + \Sigma_3. \tag{2.13}$$

We have to show that the right-hand side is actually smooth in both variables. Since

$$P_{(2)} V_{N-1}(x,y) R_\pm^{\Omega'}(2N,x)(y) = \begin{cases} C(2N,n)\, P_{(2)} V_{N-1}(x,y)\, \Gamma(x,y)^{N-n/2}, & \text{if } y \in J_\pm^{\Omega'}(x) \\ 0, & \text{otherwise} \end{cases}$$

is smooth on $(\Omega' \times \Omega') \setminus \Gamma^{-1}(0)$ and since $1-\sigma(\Gamma(x,\cdot)/\varepsilon_j)$ vanishes on a neighborhood of $\Gamma^{-1}(0)$ we have that

$$(x,y) \mapsto (1-\sigma(\Gamma(x,y)/\varepsilon_j)) \cdot P_{(2)} V_{N-1}(x,y) R_\pm^{\Omega'}(2N,x)(y)$$

is smooth. Similarly, the individual terms in the three infinite sums are smooth sections because $\sigma(\Gamma/\varepsilon_j) - \sigma(\Gamma/\varepsilon_{j+1})$, $\mathrm{grad}_{(2)}(\sigma \circ \frac{\Gamma}{\varepsilon_j})$, and $\Box_{(2)}(\sigma \circ \frac{\Gamma}{\varepsilon_j})$ all vanish on a neighborhood of $\Gamma^{-1}(0)$. It remains to be shown that the three series in (2.13) converge in all C^k-norms.

We start with Σ_2. Let $S_j := \{(x,y) \in \Omega' \times \Omega' \mid \frac{\varepsilon_j}{2} \le \Gamma(x,y) \le \varepsilon_j\}$.

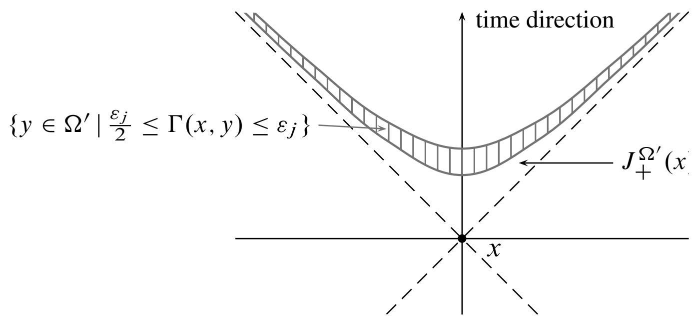


Figure 13. Section of S_j for fixed x.

Since $\mathrm{grad}_{(2)}(\sigma \circ \frac{\Gamma}{\varepsilon_j})$ vanishes outside the "strip" S_j, there exist constants $c_1(k,n)$, $c_2(k,n)$ and $c_3(k,n,j)$ such that

$$\begin{aligned}
&\Big\| \nabla^{(2)}_{\mathrm{grad}_{(2)}(\sigma\circ\frac{\Gamma}{\varepsilon_j})} \Big(V_j(\cdot,\cdot)\, R^{\Omega'}_{\pm}(2+2j,\cdot) \Big) \Big\|_{C^k(\bar\Omega\times\bar\Omega)} \\
&\quad = \Big\| \nabla^{(2)}_{\mathrm{grad}_{(2)}(\sigma\circ\frac{\Gamma}{\varepsilon_j})} \Big(V_j(\cdot,\cdot)\, R^{\Omega'}_{\pm}(2+2j,\cdot) \Big) \Big\|_{C^k(\bar\Omega\times\bar\Omega\cap S_j)} \\
&\quad \le c_1(k,n)\cdot \Big\|\sigma\circ \tfrac{\Gamma}{\varepsilon_j}\Big\|_{C^{k+1}(\bar\Omega\times\bar\Omega\cap S_j)} \cdot \Big\| V_j(\cdot,\cdot)\, R^{\Omega'}_{\pm}(2+2j,\cdot)\Big\|_{C^{k+1}(\bar\Omega\times\bar\Omega\cap S_j)} \\
&\quad \le c_2(k,n)\cdot \|\sigma\|_{C^{k+1}(\mathbb{R})} \cdot \max_{\ell=0,\dots,k+1} \Big\|\tfrac{\Gamma}{\varepsilon_j}\Big\|^{\ell}_{C^{k+1}(\bar\Omega\times\bar\Omega\cap S_j)} \\
&\qquad\quad \cdot \|V_j\|_{C^{k+1}(\bar\Omega\times\bar\Omega\cap S_j)} \cdot \|R^{\Omega'}_{\pm}(2+2j,\cdot)\|_{C^{k+1}(\bar\Omega\times\bar\Omega\cap S_j)} \\
&\quad \le c_2(k,n)\cdot \frac{1}{\varepsilon_j^{k+1}} \cdot \|\sigma\|_{C^{k+1}(\mathbb{R})} \cdot \max_{\ell=0,\dots,k+1} \|\Gamma\|^{\ell}_{C^{k+1}(\bar\Omega\times\bar\Omega)} \\
&\qquad\quad \cdot \|V_j\|_{C^{k+1}(\bar\Omega\times\bar\Omega)} \cdot \|R^{\Omega'}_{\pm}(2+2j,\cdot)\|_{C^{k+1}(\bar\Omega\times\bar\Omega\cap S_j)} \\
&\quad \le c_3(k,n,j)\cdot \frac{1}{\varepsilon_j^{k+1}} \cdot \|\sigma\|_{C^{k+1}(\mathbb{R})} \cdot \max_{\ell=0,\dots,k+1} \|\Gamma\|^{\ell}_{C^{k+1}(\bar\Omega\times\bar\Omega)} \\
&\qquad\quad \cdot \|V_j\|_{C^{k+1}(\bar\Omega\times\bar\Omega)} \cdot \|\Gamma^{1+j-n/2}\|_{C^{k+1}(\bar\Omega\times\bar\Omega\cap S_j)}.
\end{aligned}$$

By Lemma 1.1.12 we have

$$\begin{aligned}
&\|\Gamma^{1+j-n/2}\|_{C^{k+1}(\bar\Omega\times\bar\Omega\cap S_j)} \\
&\quad \le c_4(k)\cdot \|t\mapsto t^{1+j-n/2}\|_{C^{k+1}([\varepsilon_j/2,\varepsilon_j])} \cdot \max_{\ell=0,\dots,k+1} \|\Gamma\|^{\ell}_{C^{k+1}(\bar\Omega\times\bar\Omega\cap S_j)} \\
&\quad \le c_5(k,j,n)\cdot \varepsilon_j^{j-n/2-k} \cdot \max_{\ell=0,\dots,k+1} \|\Gamma\|^{\ell}_{C^{k+1}(\bar\Omega\times\bar\Omega\cap S_j)}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\Big\| \nabla^{(2)}_{\mathrm{grad}_{(2)}(\sigma\circ\frac{\Gamma}{\varepsilon_j})} \Big(V_j(\cdot,\cdot)\, R^{\Omega'}_{\pm}(2+2j,\cdot) \Big) \Big\|_{C^k(\bar\Omega\times\bar\Omega)} \\
&\quad \le c_6(k,j,n)\cdot \|\sigma\|_{C^{k+1}(\mathbb{R})} \cdot \Big(\max_{\ell=0,\dots,k+1} \|\Gamma\|^{\ell}_{C^{k+1}(\bar\Omega\times\bar\Omega)}\Big)^2 \\
&\qquad\quad \cdot \|V_j\|_{C^{k+1}(\bar\Omega\times\bar\Omega)} \cdot \varepsilon_j^{j-2k-n/2-1} \\
&\quad \le c_6(k,j,n)\cdot \|\sigma\|_{C^{k+1}(\mathbb{R})} \cdot \max_{\ell=0,\dots,k+1} \|\Gamma\|^{2\ell}_{C^{k+1}(\bar\Omega\times\bar\Omega)} \cdot \|V_j\|_{C^{k+1}(\bar\Omega\times\bar\Omega)} \cdot \varepsilon_j
\end{aligned}$$

if $j \ge 2k + n/2 + 2$. Hence if we require the (finitely many) conditions

$$c_6(k,j,n)\cdot \|V_j\|_{C^{k+1}(\bar\Omega\times\bar\Omega)} \cdot \varepsilon_j \le 2^{-j}$$

on ε_j for all $k \le j/2 - n/4 - 1$, then almost all j-th terms of the series Σ_2 are bounded in the C^k-norm by $2^{-j}\cdot \|\sigma\|_{C^{k+1}(\mathbb{R})} \cdot \max_{\ell=0,\dots,k+1} \|\Gamma\|^{2\ell}_{C^{k+1}(\bar\Omega\times\bar\Omega)}$. Thus

Σ_2 converges in the C^k-norm for any k and defines a smooth section in $E^* \boxtimes E$ over $\bar{\Omega} \times \bar{\Omega}$.

The series Σ_1 is treated similarly. To examine Σ_3 we observe that for $j \geq k + \frac{n}{2}$

$$
\begin{aligned}
&\Big\| \Big(\big(\sigma \circ \tfrac{\Gamma}{\varepsilon_j}\big) - \big(\sigma \circ \tfrac{\Gamma}{\varepsilon_{j+1}}\big) \Big) \cdot (P_{(2)} V_j) \cdot R_\pm^{\Omega'}(2+2j, \cdot) \Big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \\
&\quad \leq c_7(j,n) \cdot \Big\| \Big(\big(\sigma \circ \tfrac{\Gamma}{\varepsilon_j}\big) - \big(\sigma \circ \tfrac{\Gamma}{\varepsilon_{j+1}}\big) \Big) \cdot (P_{(2)} V_j) \cdot \Gamma^{1+j-n/2} \Big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \\
&\quad \leq c_8(k,j,n) \cdot \Big\| \Big(\big(\sigma \circ \tfrac{\Gamma}{\varepsilon_j}\big) - \big(\sigma \circ \tfrac{\Gamma}{\varepsilon_{j+1}}\big) \Big) \cdot \Gamma^{k+1} \Big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \\
&\qquad \cdot \big\| P_{(2)} V_j \big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \cdot \Big\| \Gamma^{j-k-n/2} \Big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \\
&\quad \leq c_8(k,j,n) \cdot \Big(\Big\| \big(\sigma \circ \tfrac{\Gamma}{\varepsilon_j}\big) \cdot \Gamma^{k+1} \Big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} + \Big\| \big(\sigma \circ \tfrac{\Gamma}{\varepsilon_{j+1}}\big) \cdot \Gamma^{k+1} \Big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \Big) \\
&\qquad \cdot \big\| P_{(2)} V_j \big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \cdot \Big\| \Gamma^{j-k-n/2} \Big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} . \qquad (2.14)
\end{aligned}
$$

Putting $\sigma_j(t) := \sigma(t/\varepsilon_j) \cdot t^{k+1}$ we have $\big(\sigma \circ \frac{\Gamma}{\varepsilon_j}\big) \cdot \Gamma^{k+1} = \sigma_j \circ \Gamma$. Hence by Lemmas 1.1.12 and 2.4.1

$$
\begin{aligned}
\big\| \big(\sigma \circ \tfrac{\Gamma}{\varepsilon_j}\big) \cdot \Gamma^{k+1} \big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} &= \big\| \sigma_j \circ \Gamma \big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \\
&\leq c_9(k,n) \cdot \big\| \sigma_j \big\|_{C^k(\mathbb{R})} \cdot \max_{\ell=0,\dots,k} \|\Gamma\|^\ell_{C^k(\bar{\Omega}\times\bar{\Omega})} \\
&\leq c_{10}(k,n) \cdot \varepsilon_j \cdot \|\sigma\|_{C^k(\mathbb{R})} \cdot \max_{\ell=0,\dots,k} \|\Gamma\|^\ell_{C^k(\bar{\Omega}\times\bar{\Omega})} .
\end{aligned}
$$

Plugging this into (2.14) yields

$$
\begin{aligned}
&\Big\| \Big(\big(\sigma \circ \tfrac{\Gamma}{\varepsilon_j}\big) - \big(\sigma \circ \tfrac{\Gamma}{\varepsilon_{j+1}}\big) \Big) \cdot (P_{(2)} V_j) \cdot R_\pm^{\Omega'}(2+2j, \cdot) \Big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \\
&\quad \leq c_{11}(k,j,n) \cdot (\varepsilon_j + \varepsilon_{j+1}) \cdot \|\sigma\|_{C^k(\mathbb{R})} \cdot \max_{\ell=0,\dots,k} \|\Gamma\|^\ell_{C^k(\bar{\Omega}\times\bar{\Omega})} \\
&\qquad \cdot \big\| P_{(2)} V_j \big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \cdot \big\| \Gamma^{j-k-n/2} \big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} .
\end{aligned}
$$

Hence if we add the conditions on ε_j that

$$
c_{11}(k,j,n) \cdot \varepsilon_j \cdot \big\| P_{(2)} V_j \big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \cdot \big\| \Gamma^{j-k-n/2} \big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \leq 2^{-j-1}
$$

for all $k \leq j - \frac{n}{2}$ and

$$
c_{11}(k,j-1,n) \cdot \varepsilon_j \cdot \big\| P_{(2)} V_{j-1} \big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \cdot \big\| \Gamma^{j-1-k-n/2} \big\|_{C^k(\bar{\Omega}\times\bar{\Omega})} \leq 2^{-j-2}
$$

for all $k \leq j - 1 - \frac{n}{2}$, then we have that almost all j-th terms in Σ_3 are bounded in the C^k-norm by $2^{-j} \cdot \|\sigma\|_{C^k(\mathbb{R})} \cdot \max_{\ell=0,\dots,k} \|\Gamma\|^\ell_{C^k(\bar{\Omega}\times\bar{\Omega})}$. Thus Σ_3 defines a smooth section as well. □

Lemma 2.4.4. *The ε_j in Lemmas 2.4.2 and 2.4.3 can be chosen such that in addition there is a constant $C > 0$ so that*

$$|\widetilde{\mathcal{R}}_\pm(x)[\varphi]| \le C \cdot \|\varphi\|_{C^{n+1}(\Omega)}$$

for all $x \in \overline{\Omega}$ and all $\varphi \in \mathcal{D}(\Omega, E^*)$. In particular, $\widetilde{\mathcal{R}}(x)$ is of order at most $n+1$. Moreover, the map $x \mapsto \widetilde{\mathcal{R}}_\pm(x)[\varphi]$ is for every fixed $\varphi \in \mathcal{D}(\Omega, E^*)$ a smooth section in E^*,

$$\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi] \in C^\infty(\overline{\Omega}, E^*).$$

We know already that for each $x \in \overline{\Omega}$ the distribution $\widetilde{\mathcal{R}}(x)$ is of order at most $n+1$. The point of the lemma is that the constant C in the estimate $|\widetilde{\mathcal{R}}_\pm(x)[\varphi]| \le C \cdot \|\varphi\|_{C^{n+1}(\Omega)}$ can be chosen independently of x.

Proof. Recall the definition of $\widetilde{\mathcal{R}}_\pm(x)$,

$$\widetilde{\mathcal{R}}_\pm(x) = \sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x) + \sum_{j=N}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x).$$

By Proposition 1.4.2 (10) there are constants $C_j > 0$ such that $|R_\pm^{\Omega'}(2+2j,x)[\varphi]| \le C_j \cdot \|\varphi\|_{C^{n+1}(\Omega)}$ for all φ and all $x \in \overline{\Omega}$. Thus there is a constant $C' > 0$ such that for all φ and all $x \in \overline{\Omega}$ we have

$$\Big|\sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi]\Big| \le C' \cdot \|\varphi\|_{C^{n+1}(\Omega)}.$$

The remainder term $\sum_{j=N}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y) R_\pm^{\Omega'}(2+2j,x)(y) =: f(x,y)$ is a continuous section, hence

$$\begin{aligned}\Big|\sum_{j=N}^{\infty} &\sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi]\Big| \\ &\le \|f\|_{C^0(\overline{\Omega}\times\overline{\Omega})} \cdot \mathrm{vol}(\overline{\Omega}) \cdot \|\varphi\|_{C^0(\Omega)} \\ &\le \|f\|_{C^0(\overline{\Omega}\times\overline{\Omega})} \cdot \mathrm{vol}(\overline{\Omega}) \cdot \|\varphi\|_{C^{n+1}(\Omega)}\end{aligned}$$

for all φ and all $x \in \overline{\Omega}$. Therefore $C := C' + \|f\|_{C^0(\overline{\Omega}\times\overline{\Omega})} \cdot \mathrm{vol}(\overline{\Omega})$ does the job.

To see smoothness in x we fix $k \ge 0$ and we write

$$\begin{aligned}\widetilde{\mathcal{R}}_\pm(x)[\varphi] = &\sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi] \\ &+ \sum_{j=N}^{N+k-1} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi] \\ &+ \sum_{j=N+k}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi].\end{aligned}$$

By Proposition 1.4.2 (11) the summands $\sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x)[\varphi]$ and $V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x)[\varphi]$ depend smoothly on x. By Lemma 2.4.2 the remainder $\sum_{j=N+k}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x)[\varphi]$ is C^k. Thus $x \mapsto \widetilde{\mathcal{R}}_\pm(x)[\varphi]$ is C^k for every k, hence smooth. $\square$

Definition 2.4.5. If M is a time-oriented Lorentzian manifold, then we call a subset $S \subset M \times M$ *future-stretched* with respect to M if $y \in J_+^M(x)$ whenever $(x,y) \in S$. We call it *strictly future-stretched* with respect to M if $y \in I_+^M(x)$ whenever $(x,y) \in S$. Analogously, we define *past-stretched* and *strictly past-stretched* subsets.

We summarize the results obtained so far.

Proposition 2.4.6. *Let M be an n-dimensional time-oriented Lorentzian manifold and let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let $\Omega' \subset M$ be a convex open subset. Fix an integer $N \geq \frac{n}{2}$ and fix a smooth function $\sigma : \mathbb{R} \to \mathbb{R}$ satisfying $\sigma \equiv 1$ outside $[-1,1]$, $\sigma \equiv 0$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$, and $0 \leq \sigma \leq 1$ everywhere.*

Then for every relatively compact open subset $\Omega \subset\subset \Omega'$ there exists a sequence $\varepsilon_j > 0$, $j \geq N$, such that for every $x \in \overline{\Omega}$

$$\widetilde{\mathcal{R}}_\pm(x) = \sum_{j=0}^{N-1} V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x) + \sum_{j=N}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x)$$

defines a distribution on Ω satisfying

(1) $\mathrm{supp}(\widetilde{\mathcal{R}}_\pm(x)) \subset J_\pm^{\Omega'}(x)$,

(2) $\mathrm{sing\,supp}(\widetilde{\mathcal{R}}_\pm(x)) \subset C_\pm^{\Omega'}(x)$,

(3) $P_{(2)}\widetilde{\mathcal{R}}_\pm(x) = \delta_x + K_\pm(x,\cdot)$ *with smooth* $K_\pm \in C^\infty(\overline{\Omega}\times\overline{\Omega}, E^* \boxtimes E)$,

(4) $\mathrm{supp}(K_+)$ *is future-stretched and* $\mathrm{supp}(K_-)$ *is past-stretched with respect to* Ω',

(5) $\widetilde{\mathcal{R}}_\pm(x)[\varphi]$ *depends smoothly on x for every fixed* $\varphi \in \mathcal{D}(\Omega, E^*)$,

(6) *there is a constant $C > 0$ such that* $|\widetilde{\mathcal{R}}_\pm(x)[\varphi]| \leq C \cdot \|\varphi\|_{C^{n+1}(\Omega)}$ *for all* $x \in \overline{\Omega}$ *and all* $\varphi \in \mathcal{D}(\Omega, E^*)$.

Proof. The only thing that remains to be shown is the statement (4). Recall from (2.13) that in the notation of the proof of Lemma 2.4.3

$$K_\pm(x,y) = (1-\sigma(\Gamma(x,y)/\varepsilon_{N-1}))\cdot P_{(2)}V_{N-1}(x,y)\cdot R_\pm^{\Omega'}(2N,x)(y)+\Sigma_1+\Sigma_2+\Sigma_3.$$

The first term as well as all summands in the three infinite series Σ_1, Σ_2, and Σ_3 contain a factor $R_\pm^{\Omega'}(2j,x)(y)$ for some $j \geq N$. Hence if $K_+(x,y) \neq 0$, then $y \in \mathrm{supp}(R_\pm^{\Omega'}(2j,x)) \subset J_+^{\Omega'}(x)$. In other words, $\{(x,y) \in \Omega \times \Omega \mid K_+(x,y) \neq 0\}$ is future-stretched with respect to Ω'. Since Ω' is geodesically convex causal futures are closed. Hence $\mathrm{supp}(K_+) = \overline{\{(x,y) \in \Omega\times\Omega \mid K_+(x,y) \neq 0\}}$ is future-stretched with respect to Ω' as well. In the same way one sees that $\mathrm{supp}(K_-)$ is past-stretched. $\square$

Definition 2.4.7. If the ε_j are chosen as in Proposition 2.4.6, then we call

$$\widetilde{\mathcal{R}}_\pm(x) = \sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x) + \sum_{j=N}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)$$

an *approximate advanced* or *retarded fundamental solution* respectively.

From now on we assume that $\Omega \subset\subset \Omega'$ is a relatively compact *causal* subset. Then for every $x \in \bar{\Omega}$ we have $J_\pm^{\bar{\Omega}}(x) = J_\pm^{\Omega'}(x) \cap \bar{\Omega}$. We fix approximate fundamental solutions $\widetilde{\mathcal{R}}_\pm(x)$.

We use the corresponding $K_\pm$ as an integral kernel to define an integral operator. Set for $u \in C^0(\bar{\Omega}, E^*)$ and $x \in \bar{\Omega}$

$$(\mathcal{K}_\pm u)(x) := \int_{\bar{\Omega}} K_\pm(x,y) u(y)\, \mathrm{dV}(y). \tag{2.15}$$

Since $K_\pm$ is C^∞ so is $\mathcal{K}_\pm u$, i.e., $\mathcal{K}_\pm u \in C^\infty(\bar{\Omega}, E^*)$. By the properties of the support of $K_\pm$ the integrand $K_\pm(x,y)u(y)$ vanishes unless $y \in J_\pm^{\bar{\Omega}}(x) \cap \mathrm{supp}(u)$. Hence $(\mathcal{K}_\pm u)(x) = 0$ if $J_\pm^{\bar{\Omega}}(x) \cap \mathrm{supp}(u) = \emptyset$. In other words,

$$\mathrm{supp}(\mathcal{K}_\pm u) \subset J_\mp^{\bar{\Omega}}(\mathrm{supp}(u)). \tag{2.16}$$

If we put $C_k := \int_{\bar{\Omega}} \|K_\pm(\cdot\,,y)\|_{C^k(\bar{\Omega})}\, \mathrm{dV}(y)$, then

$$\|\mathcal{K}_\pm u\|_{C^k(\bar{\Omega})} \le C_k \cdot \|u\|_{C^0(\bar{\Omega})}.$$

Hence (2.15) defines a bounded linear map

$$\mathcal{K}_\pm \colon C^0(\bar{\Omega}, E^*) \to C^k(\bar{\Omega}, E^*)$$

for all $k \ge 0$.

Lemma 2.4.8. *Let $\Omega \subset\subset \Omega'$ be causal. Suppose $\bar{\Omega}$ is so small that*

$$\mathrm{vol}(\bar{\Omega}) \cdot \|K_\pm\|_{C^0(\bar{\Omega}\times\bar{\Omega})} < 1. \tag{2.17}$$

Then

$$\mathrm{id} + \mathcal{K}_\pm \colon C^k(\bar{\Omega}, E^*) \to C^k(\bar{\Omega}, E^*)$$

is an isomorphism with bounded inverse for all $k = 0, 1, 2, \ldots$. The inverse is given by the series

$$(\mathrm{id} + \mathcal{K}_\pm)^{-1} = \sum_{j=0}^{\infty} (-\mathcal{K}_\pm)^j$$

which converges in all C^k-operator norms. The operator $(\mathrm{id} + \mathcal{K}_+)^{-1} \circ \mathcal{K}_+$ has a smooth integral kernel with future-stretched support (with respect to $\bar{\Omega}$). The operator $(\mathrm{id} + \mathcal{K}_-)^{-1} \circ \mathcal{K}_-$ has a smooth integral kernel with past-stretched support (with respect to $\bar{\Omega}$).

Proof. The operator $\mathcal{K}_\pm$ is bounded as an operator $C^0(\overline{\Omega}, E^*) \to C^k(\overline{\Omega}, E^*)$. Thus $\mathrm{id} + \mathcal{K}_\pm$ defines a bounded operator $C^k(\overline{\Omega}, E^*) \to C^k(\overline{\Omega}, E^*)$ for all k. Now

$$\begin{aligned}\|\mathcal{K}_\pm u\|_{C^0(\bar\Omega)} &\le \mathrm{vol}(\overline{\Omega}) \cdot \|K_\pm\|_{C^0(\bar\Omega\times\bar\Omega)} \cdot \|u\|_{C^0(\bar\Omega)} \\ &= (1-\eta)\cdot \|u\|_{C^0(\bar\Omega)}\end{aligned}$$

where $\eta := 1 - \mathrm{vol}(\overline{\Omega}) \cdot \|K_\pm\|_{C^0(\bar\Omega\times\bar\Omega)} > 0$. Hence the C^0-operator norm of $\mathcal{K}_\pm$ is less than 1 so that the Neumann series $\sum_{j=0}^\infty (-\mathcal{K}_\pm)^j$ converges in the C^0-operator norm and gives the inverse of $\mathrm{id} + \mathcal{K}_\pm$ on $C^0(\overline{\Omega}, E^*)$.

Next we replace the C^k-norm $\|\cdot\|_{C^k(\bar\Omega)}$ on $C^k(\overline{\Omega}, E^*)$ as defined in (1.1) by the equivalent norm

$$|||u|||_{C^k(\bar\Omega)} := \|u\|_{C^0(\bar\Omega)} + \frac{\eta}{2\,\mathrm{vol}(\overline{\Omega})\|K_\pm\|_{C^k(\bar\Omega\times\bar\Omega)} + 1}\|u\|_{C^k(\bar\Omega)}.$$

Then

$$\begin{aligned}&|||\mathcal{K}_\pm u|||_{C^k(\bar\Omega)} \\ &= \|\mathcal{K}_\pm u\|_{C^0(\bar\Omega)} + \frac{\eta}{2\,\mathrm{vol}(\overline{\Omega})\|K_\pm\|_{C^k(\bar\Omega\times\bar\Omega)} + 1}\|\mathcal{K}_\pm u\|_{C^k(\bar\Omega)} \\ &\le (1-\eta)\cdot\|u\|_{C^0(\bar\Omega)} + \frac{\eta}{2\,\mathrm{vol}(\overline{\Omega})\|K_\pm\|_{C^k(\bar\Omega\times\bar\Omega)} + 1}\,\mathrm{vol}(\overline{\Omega})\|K_\pm\|_{C^k(\bar\Omega\times\bar\Omega)}\|u\|_{C^0(\bar\Omega)} \\ &\le \left(1-\frac{\eta}{2}\right)\|u\|_{C^0(\bar\Omega)} \le \left(1-\frac{\eta}{2}\right)|||u|||_{C^k(\bar\Omega)}.\end{aligned}$$

This shows that with respect to $|||\cdot|||_{C^k(\bar\Omega)}$ the C^k-operator norm of $\mathcal{K}_\pm$ is less than 1. Thus the Neumann series $\sum_{j=0}^\infty (-\mathcal{K}_\pm)^j$ converges in all C^k-operator norms and $\mathrm{id} + \mathcal{K}_\pm$ is an isomorphism with bounded inverse on all $C^k(\overline{\Omega}, E^*)$.

For $j \ge 1$ the integral kernel of $(\mathcal{K}_\pm)^j$ is given by

$$K_\pm^{(j)}(x,y) := \int_{\bar\Omega}\cdots\int_{\bar\Omega} K_\pm(x,z_1)K_\pm(z_1,z_2)\ldots K_\pm(z_{j-1},y)\,\mathrm{dV}(z_1)\ldots\mathrm{dV}(z_{j-1}).$$

Thus $\mathrm{supp}(K_\pm^{(j)}) \subset \left\{(x,y)\in\overline{\Omega}\times\overline{\Omega} \mid y \in J_\pm^{\bar\Omega}(x)\right\}$ and

$$\begin{aligned}\|K_\pm^{(j)}\|_{C^k(\bar\Omega\times\bar\Omega)} &\le \|K_\pm\|^2_{C^k(\bar\Omega\times\bar\Omega)}\cdot\mathrm{vol}(\overline{\Omega})^{j-1}\cdot\|K_\pm\|^{j-2}_{C^0(\bar\Omega\times\bar\Omega)} \\ &\le \delta^{j-2}\cdot\mathrm{vol}(\overline{\Omega})\cdot\|K_\pm\|^2_{C^k(\bar\Omega\times\bar\Omega)}\end{aligned}$$

where $\delta := \mathrm{vol}(\overline{\Omega})\cdot\|K_\pm\|_{C^0(\bar\Omega\times\bar\Omega)} < 1$. Hence the series

$$\sum_{j=1}^\infty (-1)^{j-1}K_\pm^{(j)}$$

converges in all $C^k(\bar{\Omega} \times \bar{\Omega}, E^* \boxtimes E)$. Since this series yields the integral kernel of $(\mathrm{id} + \mathcal{K}_\pm)^{-1} \circ \mathcal{K}_\pm$ it is smooth and its support is contained in $\{(x, y) \in \bar{\Omega} \times \bar{\Omega} \mid y \in J_\pm^{\bar{\Omega}}(x)\}$. □

Corollary 2.4.9. *Let $\Omega \subset\subset \Omega'$ be as in Lemma 2.4.8. Then for each $u \in C^0(\bar{\Omega}, E)$*

$$\mathrm{supp}((\mathrm{id} + \mathcal{K}_\pm)^{-1} u) \subset J_\mp^{\bar{\Omega}}(\mathrm{supp}(u)).$$

Proof. We observe that

$$(\mathrm{id} + \mathcal{K}_\pm)^{-1} u = u - (\mathrm{id} + \mathcal{K}_\pm)^{-1} \mathcal{K}_\pm u.$$

Now $\mathrm{supp}(u) \subset J_\mp^{\bar{\Omega}}(\mathrm{supp}(u))$ and $\mathrm{supp}((\mathrm{id} + \mathcal{K}_\pm)^{-1} \mathcal{K}_\pm u) \subset J_\mp^{\bar{\Omega}}(\mathrm{supp}(u))$ by the properties of the integral kernel of $(\mathrm{id} + \mathcal{K}_\pm)^{-1} \mathcal{K}_\pm$. □

Fix $\varphi \in \mathcal{D}(\Omega, E^*)$. Then $x \mapsto \widetilde{\mathcal{R}}_\pm(x)[\varphi]$ defines a smooth section in E^* over $\bar{\Omega}$ with support contained in $J_\mp^{\Omega'}(\mathrm{supp}(\varphi)) \cap \bar{\Omega} = J_\mp^{\bar{\Omega}}(\mathrm{supp}(\varphi))$. Hence

$$F_\pm^\Omega(\cdot)[\varphi] := (\mathrm{id} + \mathcal{K}_\pm)^{-1}(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]) \tag{2.18}$$

defines a smooth section in E^* with

$$\mathrm{supp}(F_\pm^\Omega(\cdot)[\varphi]) \subset J_\mp^{\bar{\Omega}}(\mathrm{supp}(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi])) \subset J_\mp^{\bar{\Omega}}(J_\mp^{\bar{\Omega}}(\mathrm{supp}(\varphi))) = J_\mp^{\bar{\Omega}}(\mathrm{supp}(\varphi)). \tag{2.19}$$

Lemma 2.4.10. *For each $x \in \Omega$ the map $\mathcal{D}(\Omega, E^*) \mapsto E_x^*$, $\varphi \mapsto F_+^\Omega(x)[\varphi]$, is an advanced fundamental solution at x on Ω and $\varphi \mapsto F_-^\Omega(x)[\varphi]$ is a retarded fundamental solution at x on Ω.*

Proof. We first check that $\varphi \mapsto F_\pm^\Omega(x)[\varphi]$ defines a distribution for any fixed $x \in \Omega$. Let $\varphi_m \to \varphi$ in $\mathcal{D}(\Omega, E^*)$. Then $\varphi_m \to \varphi$ in $C^{n+1}(\Omega, E^*)$ and by the last point of Proposition 2.4.6 $\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi_m] \to \widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]$ in $C^0(\bar{\Omega}, E^*)$. Since $(\mathrm{id} + \mathcal{K}_\pm)^{-1}$ is bounded on C^0 we have $F_\pm^\Omega(\cdot)[\varphi_m] \to F_\pm^\Omega(\cdot)[\varphi]$ in C^0. In particular, $F_\pm^\Omega(x)[\varphi_m] \to F_\pm^\Omega(x)[\varphi]$.

Next we check that $F_\pm^\Omega(x)$ are fundamental solutions. We compute

$$\begin{aligned} P_{(2)} F_\pm^\Omega(\cdot)[\varphi] &= F_\pm^\Omega(\cdot)[P^*\varphi] \\ &= (\mathrm{id} + \mathcal{K}_\pm)^{-1}(\widetilde{\mathcal{R}}_\pm(\cdot)[P^*\varphi]) \\ &= (\mathrm{id} + \mathcal{K}_\pm)^{-1}(P_{(2)} \widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]) \\ &\overset{(2.11)}{=} (\mathrm{id} + \mathcal{K}_\pm)^{-1}(\varphi + \mathcal{K}_\pm \varphi) \\ &= \varphi. \end{aligned}$$

Thus for fixed $x \in \Omega$,

$$P F_\pm^\Omega(x)[\varphi] = \varphi(x) = \delta_x[\varphi].$$

Finally, to see that $\mathrm{supp}(F_\pm^\Omega(x)) \subset J_\pm^\Omega(x)$ let $\varphi \in \mathcal{D}(\Omega, E^*)$ such that $\mathrm{supp}(\varphi) \cap J_\pm^\Omega(x) = \emptyset$. Then $x \notin J_\mp^\Omega(\mathrm{supp}(\varphi))$ and thus $F_\pm^\Omega(x)[\varphi] = 0$ by (2.19). □

We summarize the results of this section.

Proposition 2.4.11. *Let M be a time-oriented Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let $\Omega \subset\subset M$ be a relatively compact causal domain. Suppose that Ω is sufficiently small in the sense that (2.17) holds.*

Then for each $x \in \Omega$

1. *the distributions $F_+^\Omega(x)$ and $F_-^\Omega(x)$ defined in (2.18) are fundamental solutions for P at x over Ω,*
2. $\mathrm{supp}(F_\pm^\Omega(x)) \subset J_\pm^\Omega(x)$,
3. *for each $\varphi \in \mathcal{D}(\Omega, E^*)$ the maps $x' \mapsto F_\pm^\Omega(x')[\varphi]$ are smooth sections in E^* over Ω.* □

Corollary 2.4.12. *Let M be a time-oriented Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.*

Then each point in M possesses an arbitrarily small causal neighborhood Ω such that for each $x \in \Omega$ there exist fundamental solutions $F_\pm^\Omega(x)$ for P over Ω at x. They satisfy

1. $\mathrm{supp}(F_\pm^\Omega(x)) \subset J_\pm^\Omega(x)$,
2. *for each $\varphi \in \mathcal{D}(\Omega, E^*)$ the maps $x \mapsto F_\pm^\Omega(x)[\varphi]$ are smooth sections in E^*.* □

2.5 The formal fundamental solution is asymptotic

Let M be a time-oriented Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let $\Omega' \subset M$ be a convex domain and let $\Omega \subset \Omega'$ be a relatively compact causal domain with $\overline{\Omega} \subset \Omega'$. We assume that Ω is so small that Corollary 2.4.12 applies. Using Riesz distributions and Hadamard coefficients we have constructed the formal fundamental solutions at $x \in \Omega$

$$\mathcal{R}_\pm(x) = \sum_{j=0}^{\infty} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j, x),$$

the approximate fundamental solutions

$$\widetilde{\mathcal{R}}_\pm(x) = \sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j, x) + \sum_{j=N}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j, x),$$

where $N \geq \frac{n}{2}$ is fixed, and the true fundamental solutions $F_\pm^\Omega(x)$,

$$F_\pm^\Omega(\cdot)[\varphi] = (\mathrm{id} + \mathcal{K}_\pm)^{-1}(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]).$$

The purpose of this section is to show that, in a suitable sense, the formal fundamental solution is an asymptotic expansion of the true fundamental solution. For $k \geq 0$ we define the *truncated formal fundamental solution*

$$\mathcal{R}_{\pm}^{N+k}(x) := \sum_{j=0}^{N-1+k} V_j(x,\cdot)\, R_{\pm}^{\Omega'}(2+2j,x).$$

Hence we cut the formal fundamental solution at the $(N+k)$-th term. The truncated formal fundamental solution is a well-defined distribution on Ω', $\mathcal{R}_{\pm}^{N+k}(x) \in \mathcal{D}'(\Omega', E, E_x^*)$. We will show that the true fundamental solution coincides with the truncated formal fundamental solution up to an error term which is very regular along the light cone. The larger k is, the more regular is the error term.

Proposition 2.5.1. *For every $k \in \mathbb{N}$ and every $x \in \Omega$ the difference of distributions $F_{\pm}^{\Omega}(x) - \mathcal{R}_{\pm}^{N+k}(x)$ is a C^k-section in E. In fact,*

$$(x,y) \mapsto \big(F_{\pm}^{\Omega}(x) - \mathcal{R}_{\pm}^{N+k}(x)\big)(y)$$

is of regularity C^k on $\Omega \times \Omega$.

Proof. We write

$$\big(F_{\pm}^{\Omega}(x) - \mathcal{R}_{\pm}^{N+k}(x)\big)(y) = \big(F_{\pm}^{\Omega}(x) - \widetilde{\mathcal{R}}_{\pm}(x)\big)(y) + \big(\widetilde{\mathcal{R}}_{\pm}(x) - \mathcal{R}_{\pm}^{N+k}(x)\big)(y)$$

and we show that $\big(\widetilde{\mathcal{R}}_{\pm}(x) - \mathcal{R}_{\pm}^{N+k}(x)\big)(y)$ and $\big(F_{\pm}^{\Omega}(x) - \widetilde{\mathcal{R}}_{\pm}(x)\big)(y)$ are both C^k in (x,y). Now

$$\begin{aligned}\big(\widetilde{\mathcal{R}}_{\pm}(x) - \mathcal{R}_{\pm}^{N+k}(x)\big)(y) = &\sum_{j=N}^{N+k-1} \big(\sigma(\Gamma(x,y)/\varepsilon_j) - 1\big) V_j(x,y)\, R_{\pm}^{\Omega'}(2+2j,x)(y)\\ &+ \sum_{j=N+k}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_{\pm}^{\Omega'}(2+2j,x)(y).\end{aligned}$$

From Lemma 2.4.2 we know that the infinite part

$$(x,y) \mapsto \sum_{j=N+k}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_{\pm}^{\Omega'}(2+2j,x)(y)$$

is C^k. The finite part

$$(x,y) \mapsto \sum_{j=N}^{N+k-1} \big(\sigma(\Gamma(x,y)/\varepsilon_j) - 1\big)\, V_j(x,y)\, R_{\pm}^{\Omega'}(2+2j,x)(y)$$

is actually smooth since $\sigma(\Gamma/\varepsilon_j) - 1$ vanishes on a neighborhood of $\Gamma^{-1}(0)$ which is precisely the locus where $(x,y) \mapsto R_{\pm}^{\Omega'}(2+2j,x)(y)$ is nonsmooth. Furthermore,

$$\begin{aligned}F_{\pm}^{\Omega}(\cdot)[\varphi] - \widetilde{\mathcal{R}}_{\pm}(\cdot)[\varphi] &= \big((\mathrm{id} + \mathcal{K}_{\pm})^{-1} - \mathrm{id}\big)\,(\widetilde{\mathcal{R}}_{\pm}(\cdot)[\varphi])\\ &= -\big((\mathrm{id} + \mathcal{K}_{\pm})^{-1} \circ \mathcal{K}_{\pm}\big)\,(\widetilde{\mathcal{R}}_{\pm}(\cdot)[\varphi]).\end{aligned}$$

By Lemma 2.4.8 the operator $-\big((\mathrm{id}+\mathcal{K}_\pm)^{-1}\circ\mathcal{K}_\pm\big)$ has a smooth integral kernel $L_\pm(x,y)$ whose support is future or past-stretched respectively. Hence

$$
\begin{aligned}
&F^\Omega_\pm(x)[\varphi]-\widetilde{\mathcal{R}}_\pm(x)[\varphi]\\
&\quad=\int_{\bar\Omega} L_\pm(x,y)\widetilde{\mathcal{R}}_\pm(y)[\varphi]\,\mathrm{dV}(y)\\
&\quad=\sum_{j=0}^{N-1}\int_{\bar\Omega} L_\pm(x,y)V_j(y,\cdot)\,R^{\Omega'}_\pm(2+2j,y)[\varphi]\,\mathrm{dV}(y)\\
&\qquad+\sum_{j=N}^{N+k-1}\int_{\bar\Omega} L_\pm(x,y)\sigma(\Gamma(y,\cdot)/\varepsilon_j)V_j(y,\cdot)\,R^{\Omega'}_\pm(2+2j,y)[\varphi]\,\mathrm{dV}(y)\\
&\qquad+\int_{\bar\Omega\times\bar\Omega} L_\pm(x,y)f(y,z)\varphi(z)\,\mathrm{dV}(z)\,\mathrm{dV}(y)
\end{aligned}
$$

where $f(y,z)=\sum_{j=N+k}^{\infty}\sigma(\Gamma(y,z)/\varepsilon_j)V_j(y,z)\,R^{\Omega'}_\pm(2+2j,y)(z)$ is C^k by Lemma 2.4.2. Thus $(x,z)\mapsto\int_{\bar\Omega}L_\pm(x,y)f(y,z)\,\mathrm{dV}(y)$ is a C^k-section. Write $\tilde V_j(y,z):=V_j(y,z)$ if $j\le N-1$ and $\tilde V_j(y,z):=\sigma(\Gamma(y,z)/\varepsilon_j)V_j(y,z)$ if $j\ge N$. It follows from Lemma 1.4.4

$$
\begin{aligned}
&\int_{\bar\Omega} L_\pm(x,y)\tilde V_j(y,\cdot)\,R^{\Omega'}_\pm(2+2j,y)[\varphi]\,\mathrm{dV}(y)\\
&\quad=\int_{\bar\Omega} R^{\Omega'}_\pm(2+2j,y)[z\mapsto L_\pm(x,y)\tilde V_j(y,z)\varphi(z)]\,\mathrm{dV}(y)\\
&\quad=\int_{\bar\Omega} R^{\Omega'}_\mp(2+2j,z)[y\mapsto L_\pm(x,y)\tilde V_j(y,z)\varphi(z)]\,\mathrm{dV}(z)\\
&\quad=\int_{\bar\Omega} R^{\Omega'}_\mp(2+2j,z)[y\mapsto L_\pm(x,y)\tilde V_j(y,z)]\varphi(z)\,\mathrm{dV}(z)\\
&\quad=\int_{\bar\Omega} W_j(x,z)\varphi(z)\,\mathrm{dV}(z)
\end{aligned}
$$

where $W_j(x,z)=R^{\Omega'}_\mp(2+2j,z)[y\mapsto L_\pm(x,y)\tilde V_j(y,z)]$ is smooth in (x,z) by Proposition 1.4.2 (11). Hence

$$
\big(F^\Omega_\pm(x)-\widetilde{\mathcal{R}}_\pm(x)\big)(z)=\sum_{j=0}^{N+k-1}W_j(x,z)+\int_{\bar\Omega}L_\pm(x,y)f(y,z)\,\mathrm{dV}(y)
$$

is C^k in (x,z). □

The following theorem tells us that the formal fundamental solutions are asymptotic expansions of the true fundamental solutions near the light cone.

Theorem 2.5.2. *Let M be a time-oriented Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E. Let $\Omega\subset M$ be a relatively* *compact causal domain and let* $x \in \Omega$. *Let* $F^{\Omega}_{\pm}$ *denote the fundamental solutions of* P *at* x *and* $\mathcal{R}^{N+k}_{\pm}(x)$ *the truncated formal fundamental solutions.*

Then for each $k \in \mathbb{N}$ *there exists a constant* C_k *such that*

$$\left\| \left(F^{\Omega}_{\pm}(x) - \mathcal{R}^{N+k}_{\pm}(x)\right)(y) \right\| \leq C_k \cdot |\Gamma(x,y)|^k$$

for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$.

Here $\| \cdot \|$ denotes an auxiliary norm on $E^* \boxtimes E$. The proof requires some preparation.

Lemma 2.5.3. *Let* M *be a smooth manifold. Let* $H_1, H_2 \subset M$ *be two smooth hypersurfaces globally defined by the equations* $\varphi_1 = 0$ *and* $\varphi_2 = 0$ *respectively, where* $\varphi_1, \varphi_2 \colon M \to \mathbb{R}$ *are smooth functions on* M *satisfying* $d_x\varphi_i \neq 0$ *for every* $x \in H_i$, $i = 1, 2$. *We assume that* H_1 *and* H_2 *intersect transversally.*

Let $f \colon M \to \mathbb{R}$ *be a* C^k*-function on* M, $k \in \mathbb{N}$. *Let* $k_1, k_2 \in \mathbb{N}$ *such that* $k_1 + k_2 \leq k$. *We assume that* f *vanishes to order* k_i *along* H_i, *i.e., in local coordinates* $\frac{\partial^{|\alpha|} f}{\partial x^{\alpha}}(x) = 0$ *for every* $x \in H_i$ *and every multi-index* α *with* $|\alpha| \leq k_i - 1$.

Then there exists a continuous function $F \colon M \to \mathbb{R}$ *such that*

$$f = \varphi_1^{k_1} \varphi_2^{k_2} F.$$

Proof of Lemma 2.5.3. We first prove the existence of a C^{k-k_1}-function $F_1 \colon M \to \mathbb{R}$ such that

$$f = \varphi_1^{k_1} F_1.$$

This is equivalent to saying that the function $f/\varphi_1^{k_1}$ being well defined and C^k on $M \setminus H_1$ extends to a C^{k-k_1}-function F_1 on M. Since it suffices to prove this locally, we introduce local coordinates $x^1, \dots, x^n$ so that $\varphi_1(x) = x^1$. Hence in this local chart $H_1 = \{x^1 = 0\}$.

Since $f(0, x^2, \dots, x^n) = \frac{\partial^j f}{\partial (x^1)^j}(0, x^2, \dots, x^n) = 0$ for any $(x^2, \dots, x^n)$ and $j \leq k_1 - 1$ we obtain from the Taylor expansion of f in the x^1-direction to the order $k_1 - 1$ with integral remainder term

$$f(x^1, x^2, \dots, x^n) = \int_0^{x^1} \frac{(x^1 - t)^{k_1 - 1}}{(k_1 - 1)!} \frac{\partial^{k_1} f}{\partial (x^1)^{k_1}}(t, x^2, \dots, x^n)dt.$$

In particular, for $x^1 \neq 0$

$$\begin{aligned} &f(x^1, x^2, \dots, x^n) \\ &\quad = (x^1)^{k_1-1} \int_0^{x^1} \frac{1}{(k_1 - 1)!} \left(\frac{x^1 - t}{x^1}\right)^{k_1 - 1} \frac{\partial^{k_1} f}{\partial (x^1)^{k_1}}(t, x^2, \dots, x^n)dt \\ &\quad = \frac{(x^1)^{k_1-1}}{(k_1 - 1)!} \int_0^1 (1-u)^{k_1-1} x^1 \frac{\partial^{k_1} f}{\partial (x^1)^{k_1}}(x^1 u, x^2, \dots, x^n)du \\ &\quad = \frac{(x^1)^{k_1}}{(k_1 - 1)!} \int_0^1 (1-u)^{k_1-1} \frac{\partial^{k_1} f}{\partial (x^1)^{k_1}}(x^1 u, x^2, \dots, x^n)du. \end{aligned}$$

Now $F_1(x^1,\dots,x^n) := \frac{1}{(k_1-1)!}\int_0^1 (1-u)^{k_1-1}\frac{\partial^{k_1} f}{\partial(x^1)^{k_1}}(x^1u, x^2,\dots,x^n)du$ yields a C^{k-k_1}-function because $\frac{\partial^{k_1} f}{\partial(x^1)^{k_1}}$ is C^{k-k_1}. Moreover, we have

$$f = (x^1)^{k_1}\cdot F_1 = \varphi^{k_1}\cdot F_1.$$

On $M \setminus H_1$ we have $F_1 = f/\varphi_1^{k_1}$ and so F_1 vanishes to the order k_2 on $H_2 \setminus H_1$ because f does. Since H_1 and H_2 intersect transversally the subset $H_2 \setminus H_1$ is dense in H_2. Therefore the function F_1 vanishes to the order k_2 on all of H_2. Applying the considerations above to F_1 yields a $C^{k-k_1-k_2}$-function $F\colon M \to \mathbb{R}$ such that $F_1 = \varphi_2^{k_2}\cdot F$. This concludes the proof. □

Lemma 2.5.4. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ a C^{3k+1}-function. We equip $\mathbb{R}^n$ with its standard Minkowski product $\langle\cdot\,,\cdot\rangle$ and we assume that f vanishes on all spacelike vectors.*

Then there exists a continuous function $h\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$f = h\cdot\gamma^k$$

where $\gamma(x) = -\langle x, x\rangle$.

Proof of Lemma 2.5.4. The problem here is that the hypersurface $\{\gamma = 0\}$ is the light cone which contains 0 as a singular point so that Lemma 2.5.3 does not apply directly. We will get around this difficulty by resolving the singularity.

Let $\pi\colon M := \mathbb{R}\times S^{n-1} \to \mathbb{R}^n$ be the map defined by $\pi(t,x) := tx$. It is smooth on $M = \mathbb{R}\times S^{n-1}$ and outside $\pi^{-1}(\{0\}) = \{0\}\times S^{n-1}$ it is a two-fold covering of $\mathbb{R}^n\setminus\{0\}$. The function $\widehat{f} := f\circ\pi\colon M \to \mathbb{R}$ is C^{3k+1} since f is.

Consider the following functions: $\widehat{\gamma}\colon M \to \mathbb{R}$, $\widehat{\gamma}(t,x) := \gamma(x)$, and $\pi_1\colon M \to \mathbb{R}$, $\pi_1(t,x) := t$. These functions are smooth and have only regular points on M. For $\widehat{\gamma}$ this follows from $d_x\gamma \neq 0$ for every $x \in S^{n-1}$. Therefore $\widehat{C}(0) := \widehat{\gamma}^{-1}(\{0\})$ and $\{0\}\times S^{n-1} = \pi_1^{-1}(\{0\})$ are smooth embedded hypersurfaces. Since the differentials of $\widehat{\gamma}$ and of π_1 are linearly independent the hypersurfaces intersect transversally. Furthermore, one obviously has $\pi(\widehat{C}(0)) = C(0)$ and $\pi(\{0\}\times S^{n-1}) = \{0\}$.

Since f is C^{3k+1} and vanishes on all spacelike vectors f vanishes to the order $3k+2$ along $C(0)$ (and in particular at 0). Hence $\widehat{f}$ vanishes to the order $3k+2$ along $\widehat{C}(0)$ and along $\{0\}\times S^{n-1}$. Applying Lemma 2.5.3 to $\widehat{f}$, $\varphi_1 := \pi_1$ and $\varphi_2 := \widehat{\gamma}$, with $k_1 := 2k+1$ and $k_2 := k$, yields a continuous function $\widehat{F}\colon \mathbb{R}\times S^{n-1} \to \mathbb{R}$ such that

$$\widehat{f} = \pi_1^{2k+1}\cdot\widehat{\gamma}^k\cdot\widehat{F}. \tag{2.20}$$

For $y \in \mathbb{R}^n$ we set

$$h(y) := \begin{cases} \|y\|\cdot\widehat{F}(\|y\|, \frac{y}{\|y\|}) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0, \end{cases}$$

where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^n$. The function h is obviously continuous on $\mathbb{R}^n$. It remains to show $f = \gamma^k \cdot h$. For $y \in \mathbb{R}^n \setminus \{0\}$ we have

$$\begin{aligned} f(y) &= f\left(\|y\| \cdot \frac{y}{\|y\|}\right) \\ &= \hat{f}\left(\|y\|, \frac{y}{\|y\|}\right) \\ &\overset{(2.20)}{=} \|y\|^{2k+1} \cdot \gamma\left(\frac{y}{\|y\|}\right)^k \cdot \hat{F}\left(\|y\|, \frac{y}{\|y\|}\right) \\ &= \|y\|^{2k} \cdot \gamma\left(\frac{y}{\|y\|}\right)^k \cdot h(y) \\ &= \gamma(y)^k \cdot h(y). \end{aligned}$$

For $y = 0$ the equation $f(y) = \gamma(y)^k \cdot h(y)$ holds trivially. □

Proof of Theorem 2.5.2. Repeatedly using Proposition 1.4.2 (3) we find constants C_j' such that

$$\begin{aligned} &\big(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+k}(x)\big)(y) \\ &\quad = \big(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3k+1}(x)\big)(y) + \sum_{j=N+k}^{N+3k} V_j(x,y) \cdot R_\pm^{\Omega'}(2+2j,x)(y) \\ &\quad = \big(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3k+1}(x)\big)(y) \\ &\qquad + \sum_{j=N+k}^{N+3k} V_j(x,y) \cdot C_j' \cdot \Gamma(x,y)^k \cdot R_\pm^{\Omega'}(2+2(j-k),x)(y). \end{aligned}$$

Now $h_j(x,y) := C_j' \cdot V_j(x,y) \cdot R_\pm^{\Omega'}(2+2(j-k),x)(y)$ is continuous since $2 + 2(j-k) \geq 2 + 2N \geq 2 + n > n$. By Proposition 2.5.1 the section $(x,y) \mapsto \big(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3k+1}(x)\big)(y)$ is of regularity C^{3k+1}. Moreover, we know $\operatorname{supp}(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3k+1}(x)) \subset J_\pm^\Omega(x)$. Hence we may apply Lemma 2.5.4 in normal coordinates and we obtain a continuous section h such that

$$\big(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3k+1}(x)\big)(y) = \Gamma(x,y)^k \cdot h(x,y).$$

This shows

$$\big(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+k}(x)\big)(y) = \Big(h(x,y) + \sum_{j=N+k}^{N+3k} h_j(x,y)\Big)\Gamma(x,y)^k.$$

Now $C_k := \|h + \sum_{j=N+k}^{N+3k} h_j\|_{C^0(\bar{\Omega}\times\bar{\Omega})}$ does the job. □

Remark 2.5.5. It is interesting to compare Theorem 2.5.2 to a similar situation arising in the world of Riemannian manifolds. If M is an n-dimensional compact Riemannian manifold, then the operators analogous to normally hyperbolic operators on Lorentzian manifolds are the *Laplace type* operators. They are defined formally just like normally hyperbolic operators, namely their principal symbol must be given by the metric. Analytically however, they behave very differently because they are elliptic.

If L is a nonnegative formally selfadjoint Laplace type operator on M, then it is essentially selfadjoint and one can form the semi-group $t \mapsto e^{-t\bar{L}}$ where $\bar{L}$ is the selfadjoint extension of L. For $t > 0$ the operator $e^{-t\bar{L}}$ has a smooth integral kernel $K_t(x, y)$. One can show that there is an asymptotic expansion of this "heat kernel"

$$K_t(x,x) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{k=0}^{\infty} \alpha_k(x) t^k$$

as $t \searrow 0$. The coefficients $\alpha_k(x)$ are given by a universal expression in the coefficients of L and their covariant derivatives and the curvature of M and its covariant derivatives.

Even though this asymptotic expansion is very different in nature from the one in Theorem 2.5.2, it turns out that the Hadamard coefficients on the diagonal $V_k(x, x)$ of a normally hyperbolic operator P on an n-dimensional Lorentzian manifold are *given by the same universal expression* in the coefficients of P and their covariant derivatives and the curvature of M and its covariant derivatives as $\alpha_k(x)$. This is due to the fact that the recursive relations defining α_k are formally the same as the transport equations (2.3) for P. See e.g. [Berline–Getzler–Vergne1992] for details on Laplace type operators.

2.6 Solving the inhomogeneous equation on small domains

In the next chapter we will show uniqueness of the fundamental solutions. For this we need to be able to solve the inhomogeneous equation $Pu = v$ for given v with small support. Let Ω be a relatively compact causal subset of M as in Corollary 2.4.12. Let $F^\Omega_\pm(x)$ be the corresponding fundamental solutions for P at $x \in \Omega$ over Ω. Recall that for $\varphi \in \mathcal{D}(\Omega, E^*)$ the maps $x \mapsto F^\Omega_\pm(x)[\varphi]$ are smooth sections in E^*. Using the natural pairing $E^*_x \otimes E_x \to \mathbb{K}$, $\ell \otimes e \mapsto \ell \cdot e$, we obtain a smooth $\mathbb{K}$-valued function $x \mapsto F^\Omega_\pm(x)[\varphi] \cdot v(x)$ with compact support. We put

$$u_\pm[\varphi] := \int_\Omega F^\Omega_\pm(x)[\varphi] \cdot v(x)\, \mathrm{dV}(x). \tag{2.21}$$

This defines distributions $u_\pm \in \mathcal{D}'(\Omega, E)$ because if $\varphi_m \to \varphi$ in $\mathcal{D}(\Omega, E^*)$, then $F^\Omega_\pm(\cdot)[\varphi_m] \to F^\Omega_\pm(\cdot)[\varphi]$ in $C^0(\bar{\Omega}, E^*)$ by Lemma 2.4.4 and (2.18). Hence $u_\pm[\varphi_m] \to u_\pm[\varphi]$.

Lemma 2.6.1. *The distributions $u_\pm$ defined in* (2.21) *satisfy*

$$Pu_\pm = v \quad \textit{and} \quad \mathrm{supp}(u_\pm) \subset J^\Omega_\pm(\mathrm{supp}(v)).$$

Proof. Let $\varphi \in \mathcal{D}(\Omega, E^*)$. We compute

$$
\begin{aligned}
Pu_\pm[\varphi] &= u_\pm[P^*\varphi] \\
&= \int_\Omega F_\pm^\Omega(x)[P^*\varphi] \cdot v(x)\, \mathrm{dV}(x) \\
&= \int_\Omega P_{(2)} F_\pm^\Omega(x)[\varphi] \cdot v(x)\, \mathrm{dV}(x) \\
&= \int_\Omega \varphi(x) \cdot v(x)\, \mathrm{dV}(x).
\end{aligned}
$$

Thus $Pu_\pm = v$. Now assume $\mathrm{supp}(\varphi) \cap J_\pm^\Omega(\mathrm{supp}(v)) = \emptyset$. Then $\mathrm{supp}(v) \cap J_\mp^\Omega(\mathrm{supp}(\varphi)) = \emptyset$. Since $J_\mp^\Omega(\mathrm{supp}(\varphi))$ contains the support of $x \mapsto F_\pm^\Omega(x)[\varphi]$ we have $\mathrm{supp}(v) \cap \mathrm{supp}(F_\pm^\Omega(\cdot)[\varphi]) = \emptyset$. Hence the integrand in (2.21) vanishes identically and therefore $u_\pm[\varphi] = 0$. This proves $\mathrm{supp}(u_\pm) \subset J_\pm^\Omega(\mathrm{supp}(v))$. $\square$

Lemma 2.6.2. *Let Ω be causal and contained in a convex domain Ω'. Let $S_1, S_2 \subset \Omega$ be compact subsets. Let $V \in C^\infty(\overline{\Omega} \times \overline{\Omega}, E^* \boxtimes E)$. Let $\Phi \in C^{n+1}(\overline{\Omega}, E^*)$ and $\Psi \in C^{n+1}(\overline{\Omega}, E)$ be such that* $\mathrm{supp}(\Phi) \subset J_\mp^\Omega(S_1)$ *and* $\mathrm{supp}(\Psi) \subset J_\pm^\Omega(S_2)$.

Then for all $j \geq 0$

$$
\begin{aligned}
&\int_{\overline{\Omega}} \big(V(x,\cdot) R_\pm^{\Omega'}(2+2j,x)\big)[\Phi] \cdot \Psi(x)\, \mathrm{dV}(x) \\
&\qquad = \int_{\overline{\Omega}} \Phi(y) \cdot \big(V(\cdot,y) R_\mp^{\Omega'}(2+2j,y)\big)[\Psi]\, \mathrm{dV}(y).
\end{aligned}
$$

Proof. Since $\mathrm{supp}(R_\pm^{\Omega'}(2+2j,x)) \cap \mathrm{supp}(\Phi) \subset J_\pm^\Omega(x) \cap J_\mp^\Omega(S_1)$ is compact (see Lemma A.5.7) and since the distribution $R_\pm^{\Omega'}(2+2j,x)$ is of order $\leq n+1$ we may apply $V(x,\cdot)R_\pm^{\Omega'}(2+2j,x)$ to Φ. By Proposition 1.4.2 (12) the section $x \mapsto V(x,\cdot)R_\pm^{\Omega'}(2+2j,x)[\Phi]$ is continuous. Moreover, $\mathrm{supp}(x \mapsto V(x,\cdot)R_\pm^{\Omega'}(2+2j,x)[\Phi]) \cap \mathrm{supp}(\Psi) \subset J_\mp^\Omega(\mathrm{supp}(\Phi)) \cap J_\pm^\Omega(S_2) \subset J_\mp^\Omega(S_1) \cap J_\pm^\Omega(S_2)$ is also compact and contained in $\overline{\Omega}$. Hence the integrand of the left-hand side is a compactly supported continuous function and the integral is well defined. Similarly, the integral on the right-hand side is well defined. By Lemma 1.4.3

$$
\begin{aligned}
&\int_\Omega \big(V(x,\cdot) R_\pm^{\Omega'}(2+2j,x)\big)[\Phi] \cdot \Psi(x)\, \mathrm{dV}(x) \\
&\quad = \int_\Omega R_\pm^{\Omega'}(2+2j,x)[y \mapsto V(x,y)^*\Phi(y)] \cdot \Psi(x)\ \mathrm{dV}(x) \\
&\quad = \int_\Omega R_\pm^{\Omega'}(2+2j,x)[y \mapsto \Phi(y)V(x,y)\Psi(x)]\, \mathrm{dV}(x) \\
&\quad = \int_\Omega R_\mp^{\Omega'}(2+2j,y)[x \mapsto \Phi(y)V(x,y)\Psi(x)]\, \mathrm{dV}(y) \\
&\quad = \int_\Omega \Phi(y) \cdot \big(V(\cdot,y) R_\mp^{\Omega}(2+2j,y)[\Psi]\big)\, \mathrm{dV}(y).
\end{aligned}
$$

$\square$

Lemma 2.6.3. *Let $\Omega \subset M$ be a relatively compact causal domain satisfying (2.17) in Lemma 2.4.8.*

Then the distributions $u_\pm$ defined in (2.21) are smooth sections in E, i.e., $u_\pm \in C^\infty(\Omega, E)$.

Proof. Let $\varphi \in \mathcal{D}(\Omega, E^*)$. Put $S := \operatorname{supp}(\varphi)$. Let $L_\pm \in C^\infty(\overline{\Omega} \times \overline{\Omega}, E^* \boxtimes E)$ be the integral kernel of $(\mathrm{id} + \mathcal{K}_\pm)^{-1} \circ \mathcal{K}_\pm$. We recall from (2.18)

$$F_\pm^\Omega(\cdot)[\varphi] = (\mathrm{id} + \mathcal{K}_\pm)^{-1}(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]) = \widetilde{\mathcal{R}}_\pm(\cdot)[\varphi] - (\mathrm{id} + \mathcal{K}_\pm)^{-1}\mathcal{K}_\pm(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]).$$

Therefore

$$\begin{aligned} u_\pm[\varphi] &= \int_\Omega F_\pm^\Omega(x)[\varphi] \cdot v(x)\,\mathrm{dV}(x) \\ &= \int_\Omega \widetilde{\mathcal{R}}_\pm(x)[\varphi] \cdot v(x)\,\mathrm{dV}(x) - \int_\Omega\int_\Omega L_\pm(y,x) \cdot \widetilde{\mathcal{R}}_\pm(x)[\varphi] \cdot v(y)\,\mathrm{dV}(x)\,\mathrm{dV}(y) \\ &= \int_\Omega \widetilde{\mathcal{R}}_\pm(x)[\varphi] \cdot w(x)\,\mathrm{dV}(x) \end{aligned}$$

where $w(x) := v(x) - \int_\Omega v(y) \cdot L_\pm(y,x)\,\mathrm{dV}(y) \in E_x$. Obviously, $w \in C^\infty(\overline{\Omega}, E)$. By Lemma 2.4.8 $\operatorname{supp}(L_\pm) \subset \{(y,x) \in \overline{\Omega} \times \overline{\Omega} \mid x \in J_\pm^{\overline{\Omega}}(y)\}$. Hence $\operatorname{supp}(w) \subset J_\pm^{\overline{\Omega}}(\operatorname{supp}(v))$. We may therefore apply Lemma 2.6.2 with $\Phi = \varphi$ and $\Psi = w$ to obtain

$$\int_\Omega V_j(x,\cdot) R_\pm^{\Omega'}(2+2j,x)[\varphi]\cdot w(x)\,\mathrm{dV}(x) = \int_\Omega \varphi(y) V_j(\cdot,y) R_\mp^{\Omega'}(2+2j,y)[w]\,\mathrm{dV}(y)$$

for $j = 0, \dots, N-1$ and

$$\begin{aligned} &\int_\Omega \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot) R_\pm^{\Omega'}(2+2j,x)[\varphi] \cdot w(x)\,\mathrm{dV}(x) \\ &\qquad = \int_\Omega \varphi(y)\sigma(\Gamma(\cdot,y)/\varepsilon_j) V_j(\cdot,y) R_\mp^{\Omega'}(2+2j,y)[w]\,\mathrm{dV}(y) \end{aligned}$$

for $j \geq N$. Note that the contribution of the zero set $\partial\Omega$ in the above integrals vanishes, hence we integrate over Ω instead of $\overline{\Omega}$. Summation over j yields

$$\begin{aligned} u_\pm[\varphi] &= \int_\Omega \widetilde{\mathcal{R}}_\pm(x)[\varphi] \cdot w(x)\,\mathrm{dV}(x) \\ &= \sum_{j=0}^{N-1} \int_\Omega \varphi(y) V_j(\cdot,y) R_\mp^{\Omega'}(2+2j,y)[w]\,\mathrm{dV}(y) \\ &\quad + \sum_{j=N}^{\infty} \int_\Omega \varphi(y)\sigma(\Gamma(\cdot,y)/\varepsilon_j) V_j(\cdot,y) R_\mp^{\Omega'}(2+2j,y)[w]\,\mathrm{dV}(y). \end{aligned}$$

Thus

$$u_\pm(y) = \sum_{j=0}^{N-1} \Big(V_j(\cdot\,,y) R_\mp^{\Omega'}(2+2j,y)\Big)[w] + \sum_{j=N}^{\infty} \Big(\sigma(\Gamma(\cdot\,,y)/\varepsilon_j) V_j(\cdot\,,y) R_\mp^{\Omega'}(2+2j,y)\Big)[w].$$

Proposition 1.4.2 (11) shows that all summands are smooth in y. By the choice of the ε_j the series converges in all C^k-norms. Hence $u_\pm$ is smooth. □

We summarize

Theorem 2.6.4. *Let M be a time-oriented Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.*

Then each point in M possesses a relatively compact causal neighborhood Ω such that for each $v \in \mathcal{D}(\Omega, E)$ there exist $u_\pm \in C^\infty(\Omega, E)$ satisfying

1. $\int_\Omega \varphi(x) \cdot u_\pm(x)\,\mathrm{dV} = \int_\Omega F_\pm^\Omega(x)[\varphi] \cdot v(x)\,\mathrm{dV}$ *for each* $\varphi \in \mathcal{D}(\Omega, E^*)$,
2. $Pu_\pm = v$,
3. $\mathrm{supp}(u_\pm) \subset J_\pm^\Omega(\mathrm{supp}(v))$. □

3 The global theory

In the previous chapter we developed the local theory. We proved existence of advanced and retarded fundamental solutions on small domains Ω in the Lorentzian manifold. The restriction to small domains arises from two facts. Firstly, Riesz distributions and Hadamard coefficients are defined only in domains on which the Riemannian exponential map is a diffeomorphism. Secondly, the analysis in Section 2.4 that allows us to turn the approximate fundamental solution into a true one requires sufficiently good bounds on various functions defined on Ω. Consequently, our ability to solve the wave equation as in Theorem 2.6.4 is so far also restricted to small domains.

In this chapter we will use these local results to understand solutions to a wave equation defined on the whole Lorentzian manifold. To obtain a reasonable theory we have to make geometric assumptions on the manifold. In most cases we will assume that the manifold is globally hyperbolic. This is the class of manifolds where we get a very complete understanding of wave equations.

However, in some cases we get global results for more general manifolds. We start by showing uniqueness of fundamental solutions with a suitable condition on their support. The geometric assumptions needed here are weaker than global hyperbolicity. In particular, on globally hyperbolic manifolds we get uniqueness of advanced and retarded fundamental solutions.

Then we show that the Cauchy problem is well-posed on a globally hyperbolic manifold. This means that one can uniquely solve $Pu = f$, $u|_S = u_0$ and $\nabla_{\mathfrak{n}} u = u_1$ where f, u_0 and u_1 are smooth and compactly supported, S is a Cauchy hypersurface and $\nabla_{\mathfrak{n}}$ is the covariant normal derivative along S. The solution depends continuously on the given data f, u_0 and u_1. It is unclear how one could set up a Cauchy problem on a non-globally hyperbolic manifold because one needs a Cauchy hypersurface S to impose the initial conditions $u|_S = u_0$ and $\nabla_{\mathfrak{n}} u = u_1$.

Once existence of solutions to the Cauchy problem is established it is not hard to show existence of fundamental solutions and of Green's operators. In the last section we show how one can get fundamental solutions to some operators on certain non-globally hyperbolic manifolds like anti-deSitter spacetime.

3.1 Uniqueness of the fundamental solution

The first global result is uniqueness of solutions to the wave equation with future or past compact support. For this to be true the manifold must have certain geometric properties. Recall from Definition 1.3.14 and Proposition 1.3.15 the definition and properties of the time-separation function τ. The relation "$\le$" being closed means that $p_i \le q_i$, $p_i \to p$, and $q_i \to q$ imply $p \le q$.

Theorem 3.1.1. *Let M be a connected time-oriented Lorentzian manifold such that*

(1) *the causality condition holds, i.e., there are no causal loops,*

(2) *the relation "$\le$" is closed,*

(3) *the time separation function τ is finite and continuous on $M \times M$.*

Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.

Then any distribution $u \in \mathcal{D}'(M, E)$ with past or future compact support solving the equation $Pu = 0$ must vanish identically on M,

$$u \equiv 0.$$

The idea of the proof is very simple. We would like to argue as follows: We want to show $u[\varphi] = 0$ for all test sections $\varphi \in \mathcal{D}(M, E^*)$. Without loss of generality let φ be a test section whose support is contained in a sufficiently small open subset $\Omega \subset M$ to which Theorem 2.6.4 can be applied. Solve $P^*\psi = \varphi$ in Ω. Compute

$$u[\varphi] = u[P^*\psi] \stackrel{(*)}{=} \underbrace{Pu}_{=0}[\psi] = 0.$$

The problem is that equation $(*)$ is not justified because ψ does not have compact support. The argument can be rectified in case $\mathrm{supp}(u) \cap \mathrm{supp}(\psi)$ is compact. The geometric considerations in the proof have the purpose of getting to this situation.

Proof of Theorem 3.1.1. Without loss of generality let $A := \mathrm{supp}(u)$ be future compact. We will show that A is empty. Assume the contrary and consider some $x \in A$. We fix some $y \in I_-^M(x)$. Then the intersection $A \cap J_+^M(y)$ is compact and nonempty.

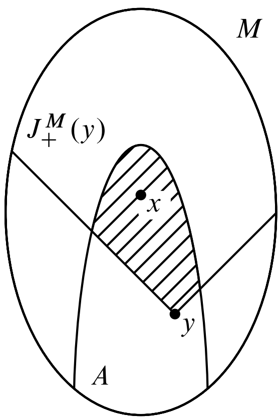


Figure 14. Uniqueness of fundamental solution; construction of y.

Since the function $M \to \mathbb{R}$, $z \mapsto \tau(y, z)$, is continuous it attains its maximum on the compact set $A \cap J_+^M(y)$ at some point $z \in A \cap J_+^M(y)$. The set $B := A \cap J_+^M(z)$ is compact and contains z. For all $z' \in B$ we have $\tau(y, z') \geq \tau(y, z)$ from (1.10) since $z' \geq z$ and hence $\tau(y, z') = \tau(y, z)$ by maximality of $\tau(y, z)$.

The relation "$\leq$" turns B into an ordered set. That $z_1 \leq z_2$ and $z_2 \leq z_1$ implies $z_1 = z_2$ follows from nonexistence of causal loops. We check that Zorn's lemma can be applied to B. Let B' be a totally ordered subset of B. Choose[1] a countable dense subset $B'' \subset B'$. Then B'' is totally ordered as well and can be written as $B'' = \{\zeta_1, \zeta_2, \zeta_3, \dots\}$. Let z_i be the largest element in $\{\zeta_1 \dots, \zeta_i\}$. This yields a monotonically increasing sequence $(z_i)_i$ which eventually becomes at least as large as any given $\zeta \in B''$.

By compactness of B a subsequence of $(z_i)_i$ converges to some $z' \in B$ as $i \to \infty$. Since the relation "$\leq$" is closed one easily sees that z' is an upper bound for B''. Since $B'' \subset B'$ is dense and "$\leq$" is closed z' is also an upper bound for B'. Hence Zorn's lemma applies and yields a maximal element $z_0 \in B$. Replacing z by z_0 we may therefore assume that $\tau(y, \cdot)$ attains its maximum at z and that $A \cap J_+^M(z) = \{z\}$.

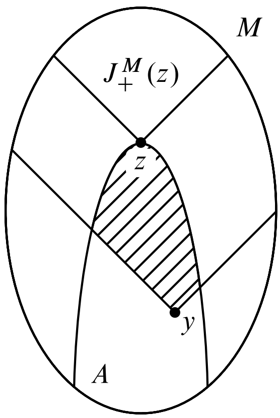


Figure 15. Uniqueness of fundamental solution; construction of z.

We fix a relatively compact causal neighborhood $\Omega \subset\subset M$ of z as in Theorem 2.6.4. (see Figure 16).

Let $p_i \in \Omega \cap I_-^M(z) \cap I_+^M(y)$ such that $p_i \to z$. We claim that for i sufficiently large we have $J_+^M(p_i) \cap A \subset \Omega$. Suppose the contrary. Then there is for each i a point $q_i \in J_+^M(p_i) \cap A$ such that $q_i \notin \Omega$. Since $q_i \in J_+^M(y) \cap A$ for all i and $J_+^M(y) \cap A$ is compact we have, after passing to a subsequence, that $q_i \to q \in J_+^M(y) \cap A$. From $q_i \geq p_i$, $q_i \to q$, $p_i \to z$, and the fact that "$\leq$" is closed we conclude $q \geq z$. Thus $q \in J_+^M(z) \cap A$, hence $q = z$. On the other hand, $q \notin \Omega$ since all $q_i \notin \Omega$, a contradiction.

[1]Every (infinite) subset of a manifold has a countable dense subset. This follows from existence of a countable basis of the topology.

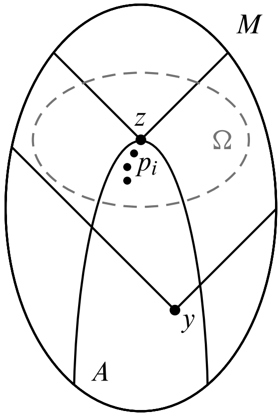


Figure 16. Uniqueness of fundamental solution; sequence $\{p_i\}_i$ converging to z.

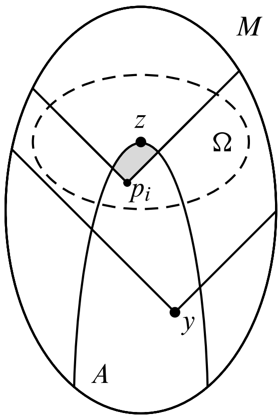


Figure 17. Uniqueness of fundamental solution; $J_+^M(p_i) \cap A \subset \Omega$ for $i \gg 0$.

This shows that we can fix i sufficiently large so that $J_+^M(p_i) \cap A \subset \Omega$. We choose a cut-off function $\eta \in \mathscr{D}(\Omega, \mathbb{R})$ such that $\eta|_{J_+^M(p_i)\cap A} \equiv 1$. We put $\widetilde{\Omega} := \Omega \cap I_+^M(p_i)$ and note that $\widetilde{\Omega}$ is an open neighborhood of z.

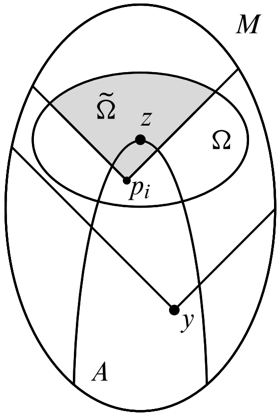


Figure 18. Uniqueness of fundamental solution; construction of the neighborhood $\tilde{\Omega}$ of z.

Now we consider some arbitrary $\varphi \in \mathcal{D}(\tilde{\Omega}, E^*)$. We will show that $u[\varphi] = 0$. This then proves that $u|_{\tilde{\Omega}} = 0$, in particular, $z \notin A = \mathrm{supp}(u)$, the desired contradiction.

By the choice of Ω we can solve the inhomogeneous equation $P^*\psi = \varphi$ on Ω with $\psi \in C^\infty(\Omega, E^*)$ and $\mathrm{supp}(\psi) \subset J_+^\Omega(\mathrm{supp}(\varphi)) \subset J_+^M(p_i) \cap \Omega$. Then $\mathrm{supp}(u) \cap \mathrm{supp}(\psi) \subset A \cap J_+^M(p_i) \cap \Omega = A \cap J_+^M(p_i)$. Hence $\eta|_{\mathrm{supp}(u)\cap\mathrm{supp}(\psi)} = 1$. Thus

$$u[\varphi] = u[P^*\psi] = u[P^*(\eta\psi)] = (Pu)[\eta\psi] = 0. \qquad \square$$

Corollary 3.1.2. *Let M be a connected time-oriented Lorentzian manifold such that*

(1) *the causality condition holds, i.e., there are no causal loops,*

(2) *the relation "$\leq$" is closed,*

(3) *the time separation function τ is finite and continuous on $M \times M$.*

Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.

Then for every $x \in M$ there exists at most one fundamental solution for P at x with past compact support and at most one with future compact support. $\square$

Remark 3.1.3. The requirement in Theorem 3.1.1 and Corollary 3.1.2 that u have future or past compact support is crucial. For example, on Minkowski space $u = R_+(2) - R_-(2)$ is a nontrivial solution to $Pu = 0$ despite the fact that Minkowski space satisfies the geometric assumptions on M in Theorem 3.1.1 and in Corollary 3.1.2.

These assumptions on M hold for convex Lorentzian manifolds and for globally hyperbolic manifolds. On a globally hyperbolic manifold the sets $J_\pm^M(x)$ are always future respectively past compact. Hence we have

Corollary 3.1.4. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.*

Then for every $x \in M$ there exists at most one advanced and at most one retarded fundamental solution for P at x. □

Remark 3.1.5. In convex Lorentzian manifolds uniqueness of advanced and retarded fundamental solutions need not hold. For example, if M is a convex open subset of Minkowski space $\mathbb{R}^n$ such that there exist points $x \in M$ and $y \in \mathbb{R}^n \setminus M$ with $J_+^{\mathbb{R}^n}(y) \cap M \subset J_+^M(x)$, then the restrictions to M of $R_+(x)$ and of $R_+(x)+R_+(y)$ are two different advanced fundamental solutions for $P = \Box$ at x on M. Corollary 3.1.2 does not apply because $J_+^M(x)$ is not past compact.

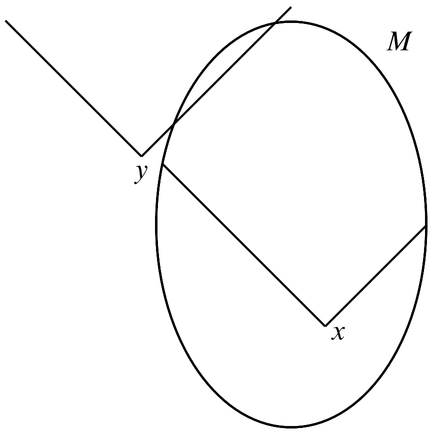


Figure 19. Advanced fundamental solution at x is not unique on M.

3.2 The Cauchy problem

The aim of this section is to show that the Cauchy problem on a globally hyperbolic manifold M is well-posed. This means that given a normally hyperbolic operator P and a Cauchy hypersurface $S \subset M$ the problem

$$\begin{aligned} Pu &= f \quad \text{on } M, \\ u &= u_0 \quad \text{along } S, \\ \nabla_{\mathfrak{n}} u &= u_1 \quad \text{along } S, \end{aligned}$$

has a unique solution for given $u_0, u_1 \in \mathcal{D}(S, E)$ and $f \in \mathcal{D}(M, E)$. Moreover, the solution depends continuously on the data.

We will also see that the support of the solution is contained in $J^M(K)$ where $K := \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)$. This is known as finiteness of propagation speed.

We start by identifying the divergence term that appears when one compares the operator P with its formal adjoint P^*. This yields a local formula allowing us to control a solution of $Pu = 0$ in terms of its Cauchy data. These local considerations already suffice to establish uniqueness of solutions to the Cauchy problem on general globally hyperbolic manifolds.

Existence of solutions is first shown locally. After some technical preparation we put these local solutions together to a global one on a globally hyperbolic manifold. This is where the crucial passage from the local to the global theory takes place. Continuous dependence of the solutions on the data is an easy consequence of the open mapping theorem from functional analysis.

Lemma 3.2.1. *Let E be a vector bundle over the time-oriented Lorentzian manifold M. Let P be a normally hyperbolic operator acting on sections in E. Let ∇ be the P-compatible connection on E.*

Then for every $\psi \in C^\infty(M, E^*)$ and $v \in C^\infty(M, E)$,

$$\psi \cdot (Pv) - (P^*\psi) \cdot v = \mathrm{div}(W),$$

where the vector field $W \in C^\infty(M, TM \otimes_{\mathbb{R}} \mathbb{K})$ is characterized by

$$\langle W, X \rangle = (\nabla_X \psi) \cdot v - \psi \cdot (\nabla_X v)$$

for all $X \in C^\infty(M, TM)$.

Here we have, as before, $\mathbb{K} = \mathbb{R}$ if E is a real vector bundle and $\mathbb{K} = \mathbb{C}$ if E is complex.

Proof. The Levi-Civita connection on TM and the P-compatible connection ∇ on E induce connections on $T^*M \otimes E$ and on $T^*M \otimes E^*$ which we also denote by ∇ for simplicity. We define a linear differential operator $L\colon C^\infty(M, T^*M \otimes E^*) \to C^\infty(M, E^*)$ of first order by

$$Ls := -\sum_{j=1}^{n} \varepsilon_j (\nabla_{e_j} s)(e_j)$$

where $e_1, \dots, e_n$ is a local Lorentz orthonormal frame of TM and $\varepsilon_j = \langle e_j, e_j \rangle$. It is easily checked that this definition does not depend on the choice of orthonormal frame. Write $e_1^*, \dots, e_n^*$ for the dual frame of T^*M. The metric $\langle \cdot, \cdot \rangle$ on TM and the natural pairing $E^* \otimes E \to \mathbb{K}$, $\psi \otimes v \mapsto \psi \cdot v$, induce a pairing $(T^*M \otimes E^*) \otimes (T^*M \otimes E) \to \mathbb{K}$ which we again denote by $\langle \cdot, \cdot \rangle$. For all $\psi \in C^\infty(M, E^*)$ and $s \in C^\infty(M, T^*M \otimes E)$

we obtain

$$\begin{aligned}
\langle \nabla\psi, s\rangle &= \sum_{j,k=1}^{n} \langle e_j^* \otimes \nabla_{e_j}\psi \,,\, e_k^* \otimes s(e_k)\rangle \\
&= \sum_{j,k=1}^{n} \langle e_j^* \,,\, e_k^*\rangle \cdot (\nabla_{e_j}\psi) \cdot s(e_k) \\
&= \sum_{j=1}^{n} \varepsilon_j (\nabla_{e_j}\psi) \cdot s(e_j) \\
&= \sum_{j=1}^{n} \varepsilon_j \left(\partial_{e_j}(\psi \cdot s(e_j)) - \psi \cdot (\nabla_{e_j} s)(e_j) - \psi \cdot s(\nabla_{e_j} e_j)\right) \\
&= \psi \cdot (Ls) + \sum_{j=1}^{n} \varepsilon_j \left(\partial_{e_j}(\psi \cdot s(e_j)) - \psi \cdot s(\nabla_{e_j} e_j)\right). \qquad (3.1)
\end{aligned}$$

Let V_1 be the unique $\mathbb{K}$-valued vector field characterized by $\langle V_1, X\rangle = \psi \cdot s(X)$ for every $X \in C^\infty(M, TM)$. Then

$$\begin{aligned}
\operatorname{div}(V_1) &= \sum_{j=1}^{n} \varepsilon_j \langle \nabla_{e_j} V_1 \,,\, e_j\rangle \\
&= \sum_{j=1}^{n} \varepsilon_j \left(\partial_{e_j}\langle V_1 \,,\, e_j\rangle - \langle V_1 \,,\, \nabla_{e_j} e_j\rangle\right) \\
&= \sum_{j=1}^{n} \varepsilon_j \left(\partial_{e_j}(\psi \cdot s(e_j)) - \psi \cdot s(\nabla_{e_j} e_j)\right).
\end{aligned}$$

Plugging this into (3.1) yields

$$\langle \nabla\psi, s\rangle = \psi \cdot Ls + \operatorname{div}(V_1).$$

In particular, if $v \in C^\infty(M, E)$ we get for $s := \nabla v \in C^\infty(M, T^*M \otimes E)$

$$\langle \nabla\psi, \nabla v\rangle = \psi \cdot L\nabla v + \operatorname{div}(V_1) = \psi \cdot \Box^\nabla v + \operatorname{div}(V_1),$$

hence

$$\psi \cdot \Box^\nabla v = \langle \nabla\psi, \nabla v\rangle - \operatorname{div}(V_1) \qquad (3.2)$$

where $\langle V_1, X\rangle = \psi \cdot \nabla_X v$ for all $X \in C^\infty(M, TM)$. Similarly, we obtain

$$(\Box^\nabla \psi) \cdot v = \langle \nabla\psi, \nabla v\rangle - \operatorname{div}(V_2)$$

where V_2 is the vector field characterized by $\langle V_2, X\rangle = (\nabla_X \psi) \cdot v$ for all $X \in C^\infty(M, TM)$. Thus

$$\psi \cdot \Box^\nabla v = (\Box^\nabla \psi) \cdot v - \operatorname{div}(V_1) + \operatorname{div}(V_2) = (\Box^\nabla \psi) \cdot v + \operatorname{div}(W)$$

where $W = V_2 - V_1$. Since ∇ is the P-compatible connection on E we have $P = \Box^\nabla + B$ for some $B \in C^\infty(M, \mathrm{End}(E))$, see Lemma 1.5.5. Thus

$$\psi \cdot Pv = \psi \cdot \Box^\nabla v + \psi \cdot Bv = (\Box^\nabla \psi) \cdot v + \mathrm{div}(W) + (B^*\psi) \cdot v.$$

If ψ or v has compact support, then we can integrate $\psi \cdot Pv$ and the divergence term vanishes. Therefore

$$\int_M \psi \cdot Pv \, \mathrm{dV} = \int_M \big((\Box^\nabla \psi) \cdot v + (B^*\psi) \cdot v\big) \, \mathrm{dV}.$$

Thus $\Box^\nabla \psi + B^*\psi = P^*\psi$ and $\psi \cdot Pv = P^*\psi \cdot v + \mathrm{div}(W)$ as claimed. □

Lemma 3.2.2. *Let E be a vector bundle over a time-oriented Lorentzian manifold M and let P be a normally hyperbolic operator acting on sections in E. Let ∇ be the P-compatible connection on E. Let $\Omega \subset M$ be a relatively compact causal domain satisfying the conditions of Lemma 2.4.8. Let S be a smooth spacelike Cauchy hypersurface in Ω. Denote by $\mathfrak{n}$ the future directed (timelike) unit normal vector field along S.*

For every $x \in \Omega$ let $F_\pm^\Omega(x)$ be the fundamental solution for P^* at x with support in $J_\pm^\Omega(x)$ constructed in Proposition 2.4.11.

Let $u \in C^\infty(\Omega, E)$ be a solution of $Pu = 0$ on Ω. Set $u_0 := u_{|S}$ and $u_1 := \nabla_\mathfrak{n} u$. Then for every $\varphi \in \mathcal{D}(\Omega, E^*)$,

$$\int_\Omega \varphi \cdot u \, \mathrm{dV} = \int_S \big((\nabla_\mathfrak{n}(F^\Omega[\varphi])) \cdot u_0 - (F^\Omega[\varphi]) \cdot u_1\big) \, \mathrm{dA},$$

where $F^\Omega[\varphi] \in C^\infty(\Omega, E^*)$ is defined as a distribution by

$$(F^\Omega[\varphi])[w] := \int_\Omega \varphi(x) \cdot (F_+^\Omega(x)[w] - F_-^\Omega(x)[w]) \, \mathrm{dV}(x)$$

for every $w \in \mathcal{D}(\Omega, E)$.

Proof. Fix $\varphi \in \mathcal{D}(\Omega, E^*)$. We consider the distribution ψ defined by $\psi[w] := \int_\Omega \varphi(x) \cdot F_+^\Omega(x)[w] \, \mathrm{dV}$ for every $w \in \mathcal{D}(\Omega, E)$. By Theorem 2.6.4 we know that $\psi \in C^\infty(\Omega, E^*)$, has its support contained in $J_+^\Omega(\mathrm{supp}(\varphi))$ and satisfies $P^*\psi = \varphi$.

Let W be the vector field from Lemma 3.2.1 with u instead of v. Since by Corollary A.5.4 the subset $J_+^\Omega(\mathrm{supp}(\varphi)) \cap J_-^\Omega(S)$ of Ω is compact, Theorem 1.3.16 applies to $D := I_-^\Omega(S)$ and the vector field W:

$$\begin{aligned}
\int_D \big((P^*\psi) \cdot u - \psi \cdot (Pu)\big) \, \mathrm{dV} &= -\int_D \mathrm{div}(W) \, \mathrm{dV} \\
&= -\underbrace{\langle \mathfrak{n}, \mathfrak{n} \rangle}_{=-1} \int_{\partial D} \langle W, \mathfrak{n} \rangle \, \mathrm{dA} \\
&= \int_{\partial D} \big((\nabla_\mathfrak{n}\psi) \cdot u - \psi \cdot (\nabla_\mathfrak{n} u)\big) \, \mathrm{dA} \\
&= \int_S \big((\nabla_\mathfrak{n}\psi) \cdot u - \psi \cdot (\nabla_\mathfrak{n} u)\big) \, \mathrm{dA}\,.
\end{aligned}$$

On the other hand,

$$\int_D \big((P^*\psi)\cdot u - \psi\cdot(Pu)\big)\,\mathrm{dV} = \int_{I_-^\Omega(S)} \big((\underbrace{P^*\psi}_{=\varphi})\cdot u - \psi\cdot(\underbrace{Pu}_{=0})\big)\,\mathrm{dV}$$
$$= \int_{I_-^\Omega(S)} \varphi\cdot u\,\mathrm{dV}.$$

Thus

$$\int_{I_-^\Omega(S)} \varphi\cdot u\,\mathrm{dV} = \int_S \big((\nabla_{\mathfrak{n}}\psi)\cdot u - \psi\cdot(\nabla_{\mathfrak{n}}u)\big)\,\mathrm{dA}\,. \tag{3.3}$$

Similarly, using $D = I_+^\Omega(S)$ and $\psi'[w] := \int_\Omega \varphi(x)\cdot F_-^\Omega(x)[w]\,\mathrm{dV}$ for any $w \in \mathcal{D}(\Omega, E)$ one gets

$$\int_{I_+^\Omega(S)} \varphi\cdot u\,\mathrm{dV} = \int_S \big(\psi'\cdot(\nabla_{\mathfrak{n}}u) - (\nabla_{\mathfrak{n}}\psi')\cdot u\big)\,\mathrm{dA}\,. \tag{3.4}$$

The different sign is caused by the fact that $\mathfrak{n}$ is the *interior* unit normal to $I_+^\Omega(S)$. Adding (3.3) and (3.4) we get

$$\int_\Omega \varphi\cdot u\,\mathrm{dV} = \int_S \big((\nabla_{\mathfrak{n}}(\psi-\psi'))\cdot u - (\psi-\psi')\cdot(\nabla_{\mathfrak{n}}u)\big)\,\mathrm{dA},$$

which is the desired result. □

Corollary 3.2.3. *Let Ω, u, u_0, and u_1 be as in Lemma 3.2.2. Then*

$$\mathrm{supp}(u) \subset J^\Omega(K)$$

where $K = \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1)$.

Proof. Let $\varphi \in \mathcal{D}(\Omega, E^*)$. From Theorem 2.6.4 we know that $\mathrm{supp}(F^\Omega[\varphi]) \subset J^\Omega(\mathrm{supp}(\varphi))$. Hence if, under the hypotheses of Lemma 3.2.2,

$$\mathrm{supp}(u_j) \cap J^\Omega(\mathrm{supp}(\varphi)) = \emptyset \tag{3.5}$$

for $j = 0, 1$, then $\int_\Omega \varphi\cdot u\,\mathrm{dV} = 0$. Equation (3.5) is equivalent to

$$\mathrm{supp}(\varphi) \cap J^\Omega(\mathrm{supp}(u_j)) = \emptyset.$$

Thus $\int_\Omega \varphi\cdot u\,\mathrm{dV} = 0$ whenever the support of the test section φ is disjoint from $J^\Omega(K)$. We conclude that u must vanish outside $J^\Omega(K)$. □

Corollary 3.2.4. *Let E be a vector bundle over a globally hyperbolic Lorentzian manifold M. Let ∇ be a connection on E and let $P = \Box^\nabla + B$ be a normally hyperbolic operator acting on sections in E. Let S be a smooth spacelike Cauchy hypersurface in M, and let $\mathfrak{n}$ be the future directed (timelike) unit normal vector field along S.*

If $u \in C^\infty(M, E)$ *solves*

$$\begin{aligned} Pu &= 0 \quad \textit{on } M, \\ u &= 0 \quad \textit{along } S, \\ \nabla_{\mathfrak{n}} u &= 0 \quad \textit{along } S, \end{aligned}$$

then $u = 0$ *on* M.

Proof. By Theorem 1.3.13 there is a foliation of M by spacelike smooth Cauchy hypersurfaces S_t ($t \in \mathbb{R}$) with $S_0 = S$. Extend $\mathfrak{n}$ smoothly to all of M such that $\mathfrak{n}_{|S_t}$ is the unit future directed (timelike) normal vector field on S_t for every $t \in \mathbb{R}$. Let $p \in M$. We show that $u(p) = 0$.

Let $T \in \mathbb{R}$ be such that $p \in S_T$. Without loss of generality let $T > 0$ and let p be in the causal future of S. Set

$$t_0 := \sup \left\{ t \in [0, T] \mid u \text{ vanishes on } J_-^M(p) \cap \left(\textstyle\bigcup_{0 \le \tau \le t} S_\tau \right) \right\}.$$

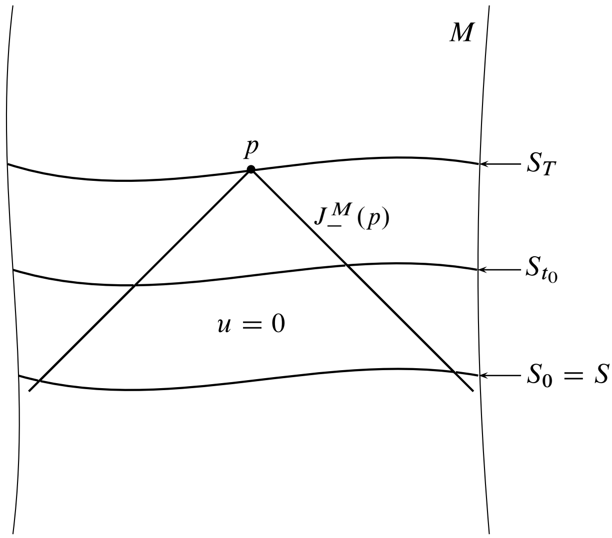


Figure 20. Uniqueness of solution to Cauchy problem; domain where u vanishes.

We will show that $t_0 = T$ which implies in particular $u(p) = 0$.

Assume $t_0 < T$. For each $x \in J_-^M(p) \cap S_{t_0}$ we may, according to Lemma A.5.6, choose a relatively compact causal neighborhood Ω of x in M satisfying the hypotheses of Lemma 2.4.8 and such that $S_{t_0} \cap \Omega$ is a Cauchy hypersurface of Ω.

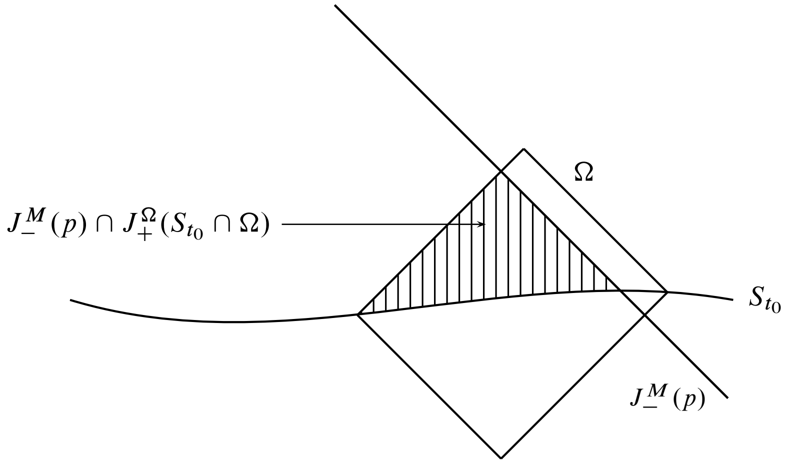


Figure 21. Uniqueness of solution to Cauchy problem; u vanishes on $J_-^M(p) \cap J_+^\Omega(S_{t_0} \cap \Omega)$.

Put $u_0 := u_{|S_{t_0}}$ and $u_1 := (\nabla_\mathfrak{n} u)_{|S_{t_0}}$. If $t_0 = 0$, then $u_0 = u_1 = 0$ on $S = S_0$ by assumption. If $t_0 > 0$, then $u_0 = u_1 = 0$ on $S_{t_0} \cap J_-^M(p)$ because $u \equiv 0$ on $J_-^M(p) \cap \left(\bigcup_{0 \le \tau \le t} S_\tau\right)$. Corollary 3.2.3 implies $u = 0$ on $J_-^M(p) \cap J_+^\Omega(S_{t_0} \cap \Omega)$.

By Corollary A.5.4 the intersection $S_{t_0} \cap J_-^M(p)$ is compact. Hence it can be covered by finitely many open subsets Ω_i, $1 \le i \le N$, satisfying the conditions of Ω

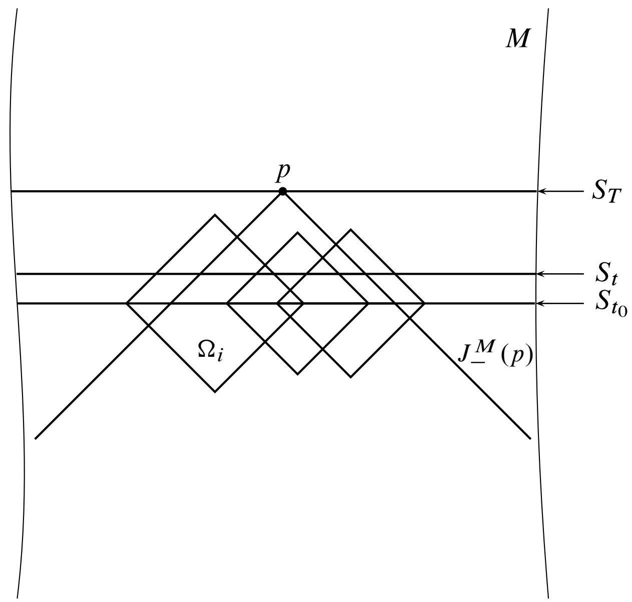


Figure 22. Uniqueness of solution to Cauchy problem; $S_t \cap J_-^M(p)$ is contained in $\bigcup_i \Omega_i$ for $t \in [t_0, t_0 + \varepsilon)$.

above. Thus u vanishes identically on $(\Omega_1 \cup \cdots \cup \Omega_N) \cap J_-^M(p) \cap J_+^M(S_{t_0})$. Since $(\Omega_1 \cup \cdots \cup \Omega_N) \cap J_-^M(p)$ is an open neighborhood of the compact set $S_{t_0} \cap J_-^M(p)$ in $J_-^M(p)$ there exists an $\varepsilon > 0$ such that $S_t \cap J_-^M(p) \subset \Omega_1 \cup \cdots \cup \Omega_N$ for every $t \in [t_0, t_0 + \varepsilon)$.

Hence u vanishes on $S_t \cap J_-^M(p)$ for all $t \in [t_0, t_0 + \varepsilon)$. This contradicts the maximality of t_0. □

Next we prove existence of solutions to the Cauchy problem on small domains. Let $\Omega \subset M$ satisfy the hypotheses of Lemma 2.4.8. In particular, Ω is relatively compact, causal, and has "small volume". Such domains will be referred to as *RCCSV* (for "Relatively Compact Causal with Small Volume"). Note that each point in a Lorentzian manifold possesses a basis of RCCSV-neighborhoods. Since causal domains are contained in convex domains by definition and convex domains are contractible, the vector bundle E is trivial over any RCCSV-domain Ω. We shall show that one can uniquely solve the Cauchy problem on every RCCSV-domain with Cauchy data on a fixed Cauchy hypersurface in Ω.

Proposition 3.2.5. *Let M be a time-oriented Lorentzian manifold and let $S \subset M$ be a spacelike hypersurface. Let $\mathfrak{n}$ be the future directed timelike unit normal field along S.*

Then for each RCCSV-domain $\Omega \subset M$ such that $S \cap \Omega$ is a (spacelike) Cauchy hypersurface in Ω, the following holds:

For each $u_0, u_1 \in \mathcal{D}(S \cap \Omega, E)$ and for each $f \in \mathcal{D}(\Omega, E)$ there exists a unique $u \in C^\infty(\Omega, E)$ satisfying

$$\begin{aligned} Pu &= f \quad \text{on } M, \\ u &= u_0 \quad \text{along } S, \\ \nabla_{\mathfrak{n}} u &= u_1 \quad \text{along } S. \end{aligned}$$

Moreover, $\mathrm{supp}(u) \subset J^M(K)$ *where* $K = \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)$.

Proof. Let $\Omega \subset M$ be an RCCSV-domain such that $S \cap \Omega$ is a Cauchy hypersurface in Ω. Corollary 3.2.4 can then be applied on Ω: If u and $\tilde{u}$ are two solutions of the Cauchy problem, then $P(u - \tilde{u}) = 0$, $(u - \tilde{u})|_S = 0$, and $\nabla_{\mathfrak{n}}(u - \tilde{u}) = 0$. Corollary 3.2.4 implies $u - \tilde{u} = 0$ which shows uniqueness. It remains to show existence.

Since causal domains are globally hyperbolic we may apply Theorem 1.3.13 and find an isometry $\Omega = \mathbb{R} \times (S \cap \Omega)$ where the metric takes the form $-\beta dt^2 + g_t$. Here $\beta\colon \Omega \to \mathbb{R}_+^*$ is smooth, each $\{t\} \times (S \cap \Omega)$ is a smooth spacelike Cauchy hypersurface in Ω, and $S \cap \Omega$ corresponds to $\{0\} \times (S \cap \Omega)$. Note that the future directed unit normal vector field $\mathfrak{n}$ along $\{t\} \times (S \cap \Omega)$ is given by $\mathfrak{n}(\cdot) = \frac{1}{\sqrt{\beta(t,\cdot)}} \frac{\partial}{\partial t}$.

Now let $u_0, u_1 \in \mathcal{D}(S \cap \Omega, E)$ and $f \in \mathcal{D}(\Omega, E)$. We trivialize the bundle E over Ω and identify sections in E with $\mathbb{K}^r$-valued functions where r is the rank of E.

Assume for a moment that u were a solution to the Cauchy problem of the form $u(t,x) = \sum_{j=0}^{\infty} t^j u_j(x)$ where $x \in S \cap \Omega$. Write $P = \frac{1}{\beta}\frac{\partial^2}{\partial t^2} + Y$ where Y is a differential operator containing t-derivatives only up to order 1. Equation

$$f = Pu = \left(\frac{1}{\beta}\frac{\partial^2}{\partial t^2} + Y\right)u = \frac{1}{\beta(t,\cdot)}\sum_{j=2}^{\infty} j(j-1)t^{j-2}u_j + Yu \tag{3.6}$$

evaluated at $t = 0$ gives

$$\frac{2}{\beta(0,x)}u_2(x) = -Y(u_0 + tu_1)(0,x) + f(0,x)$$

for every $x \in S \cap \Omega$. Thus u_2 is determined by u_0, u_1, and $f|_S$. Differentiating (3.6) with respect to $\frac{\partial}{\partial t}$ and repeating the procedure shows that each u_j is recursively determined by $u_0, \dots, u_{j-1}$ and the normal derivatives of f along S.

Now we drop the assumption that we have a t-power series u solving the problem but we *define* the u_j, $j \geq 2$, by these recursive relations. Then $\mathrm{supp}(u_j) \subset \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup (\mathrm{supp}(f) \cap S)$ for all j.

Let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\sigma|_{[-1/2,1/2]} \equiv 1$ and $\sigma \equiv 0$ outside $[-1,1]$. We claim that we can find a sequence of $\varepsilon_j \in (0,1)$ such that

$$\hat{u}(t,x) := \sum_{j=0}^{\infty} \sigma(t/\varepsilon_j)t^j u_j(x) \tag{3.7}$$

defines a smooth section that can be differentiated termwise.

By Lemma 1.1.11 we have for $j > k$

$$\|\sigma(t/\varepsilon_j)t^j u_j(x)\|_{C^k(\Omega)} \leq c(k) \cdot \left\|\sigma(t/\varepsilon_j)t^j\right\|_{C^k(\mathbb{R})} \cdot \|u_j\|_{C^k(S)}.$$

Here and in the following $c(k)$, $c'(k,j)$, and $c''(k,j)$ denote universal constants depending only on k and j. By Lemma 2.4.1 we have for $l \leq k$ and $0 < \varepsilon_j \leq 1$

$$\left\|\frac{d^l}{dt^l}(\sigma(t/\varepsilon_j)t^j)\right\|_{C^0(\mathbb{R})} \leq \varepsilon_j\, c'(l,j)\, \|\sigma\|_{C^l(\mathbb{R})},$$

thus

$$\|\sigma(t/\varepsilon_j)t^j u_j(x)\|_{C^k(\Omega)} \leq \varepsilon_j\, c''(k,j)\, \|\sigma\|_{C^k(\mathbb{R})}\, \|u_j\|_{C^k(S)}.$$

Now we choose $0 < \varepsilon_j \leq 1$ so that $\varepsilon_j\, c''(k,j)\, \|\sigma\|_{C^k(\mathbb{R})}\, \|u_j\|_{C^k(S)} \leq 2^{-j}$ for all $k < j$. Then the series (3.7) defining $\hat{u}$ converges absolutely in the C^k-norm for all k. Hence $\hat{u}$ is a smooth section with compact support and can be differentiated termwise. From the construction of $\hat{u}$ one sees that $\mathrm{supp}(\hat{u}) \subset J^M(K)$.

By the choice of the u_j the section $P\hat{u} - f$ vanishes to infinite order along S. Therefore

$$w(t,x) := \begin{cases} (P\hat{u} - f)(t,x), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases}$$

defines a smooth section with compact support. By Theorem 2.6.4 (which can be applied since the hypotheses of Lemma 2.4.8 are fulfilled) we can solve the equation $P\tilde{u} = w$ with a smooth section $\tilde{u}$ having past compact support. Moreover, $\mathrm{supp}(\tilde{u}) \subset J_+^M(\mathrm{supp}(w)) \subset J_+^M(\mathrm{supp}(\hat{u}) \cup \mathrm{supp}(f)) \subset J^M(K)$.

Now $u_+ := \hat{u} - \tilde{u}$ is a smooth section such that $Pu_+ = P\hat{u} - P\tilde{u} = w + f - w = f$ on $J_+^\Omega(S \cap \Omega) = \{t \geq 0\}$.

The restriction of $\tilde{u}$ to $I_-^\Omega(S)$ has past compact support and satisfies $P\tilde{u} = 0$ on $I_-^\Omega(S)$, thus by Theorem 3.1.1 $\tilde{u} = 0$ on $I_-^\Omega(S)$. Thus u_+ coincides with $\hat{u}$ to infinite order along S. In particular, $u_+|_S = \tilde{u}|_S = u_0$ and $\nabla_\mathfrak{n} u_+ = \nabla_\mathfrak{n} \tilde{u} = u_1$. Moreover, $\mathrm{supp}(u_+) \subset \mathrm{supp}(\hat{u}) \cup \mathrm{supp}(\tilde{u}) \subset J^M(K)$. Thus u_+ has all the required properties on $J_+^M(S)$.

Similarly, one constructs u_- on $J_-^M(S)$. Since both u_+ and u_- coincide to infinite order with $\hat{u}$ along S we obtain the smooth solution by setting

$$u(t,x) := \begin{cases} u_+(t,x), & \text{if } t \geq 0, \\ u_-(t,x), & \text{if } t \leq 0. \end{cases} \qquad \square$$

Remark 3.2.6. It follows from Lemma A.5.6 that every point p on a spacelike hypersurface S possesses a RCCSV-neighborhood Ω such that $S \cap \Omega$ is a Cauchy hypersurface in Ω. Hence Proposition 3.2.5 guarantees the local existence of solutions to the Cauchy problem.

In order to show existence of solutions to the Cauchy problem on globally hyperbolic manifolds we need some preparation. Let M be globally hyperbolic. We write $M = \mathbb{R} \times S$ and suppose the metric is of the form $-\beta dt^2 + g_t$ as in Theorem 1.3.10. Hence M is foliated by the smooth spacelike Cauchy hypersurfaces $\{t\} \times S =: S_t$, $t \in \mathbb{R}$. Let $p \in M$. Then there exists a unique t such that $p \in S_t$. For any $r > 0$ we denote by $B_r(p)$ the open ball *in* S_t of radius r about p with respect to the *Riemannian* metric g_t on S_t. Then $B_r(p)$ is open as a subset of S_t but not as a subset of M.

Recall that $D(A)$ denotes the Cauchy development of a subset A of M (see Definition 1.3.5).

Lemma 3.2.7. *The function* $\rho\colon M \to (0, \infty]$ *defined by*

$$\rho(p) := \sup\{r > 0 \mid D(B_r(p)) \textit{ is RCCSV}\},$$

is lower semi-continuous on M.

Proof. First note that ρ is well defined since every point has a RCCSV-neighborhood. Let $p \in M$ and $r > 0$ be such that $\rho(p) > r$. Let $\varepsilon > 0$. We want to show $\rho(p') > r - \varepsilon$ for all p' in a neighborhood of p.

For any point $p' \in D(B_r(p))$ consider

$$\lambda(p') := \sup\{r' > 0 \mid B_{r'}(p') \subset D(B_r(p))\}.$$

Claim: There exists a neighborhood V of p such that for every $p' \in V$ one has $\lambda(p') > r - \varepsilon$.

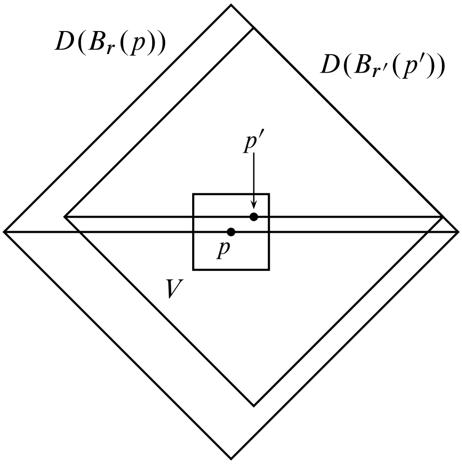


Figure 23. Construction of the neighborhood V of p.

Let us assume the claim for a moment. Let $p' \in V$. Pick r' with $r-\varepsilon < r' < \lambda(p')$. Hence $B_{r'}(p') \subset D(B_r(p))$. By Remark 1.3.6 we know $D(B_{r'}(p')) \subset D(B_r(p))$. Since $D(B_r(p))$ is RCCSV the subset $D(B_{r'}(p'))$ is RCCSV as well. Thus $\rho(p') \geq r' > r - \varepsilon$. This then concludes the proof.

It remains to show the claim. Assume the claim is false. Then there is a sequence $(p_i)_i$ of points in M converging to p such that $\lambda(p_i) \leq r - \varepsilon$ for all i. Hence for $r' := r - \varepsilon/2$ we have $B_{r'}(p_i) \not\subset D(B_r(p))$. Choose $x_i \in B_{r'}(p_i) \setminus D(B_r(p))$.

The closed set $\bar{B}_r(p)$ is contained in the compact set $\bar{D}(B_r(p))$ and therefore compact itself. Thus $[-1,1] \times \bar{B}_r(p)$ is compact. For i sufficiently large $B_{r'}(p_i) \subset [-1,1] \times \bar{B}_r(p)$ and therefore $x_i \in [-1,1] \times \bar{B}_r(p)$. We pass to a convergent subsequence $x_i \to x$. Since $p_i \to p$ and $x_i \in \bar{B}_{r'}(p_i)$ we have $x \in \bar{B}_{r'}(p)$. Hence $x \in B_r(p)$. Since $D(B_r(p))$ is an open neighborhood of x we must have $x_i \in D(B_r(p))$ for sufficiently large i. This contradicts the choice of the x_i. $\square$

For every $r > 0$ and $q \in M = \mathbb{R} \times S$ consider

$$\theta_r(q) := \sup\{\eta > 0 \mid J^M(\bar{B}_{r/2}(q)) \cap ([t_0 - \eta, t_0 + \eta] \times S) \subset D(B_r(q))\}.$$

Remark 3.2.8. There exist $\eta > 0$ with $J^M(\bar{B}_{r/2}(q)) \cap ([t_0 - \eta, t_0 + \eta] \times S) \subset D(B_r(q))$. Hence $\theta_r(q) > 0$.

One can see this as follows. If no such η existed, then there would be points $x_i \in J^M(\bar{B}_{r/2}(q)) \cap \left(\left[t_0 - \frac{1}{i}, t_0 + \frac{1}{i}\right] \times S\right)$ but $x_i \notin D(B_r(q))$, $i \in \mathbb{N}$. All x_i lie in the compact set $J^M(\bar{B}_{r/2}(q)) \cap ([t_0 - 1, t_0 + 1] \times S)$. Hence we may pass to a convergent subsequence $x_i \to x$. Then $x \in J^M(\bar{B}_{r/2}(q)) \cap (\{t_0\} \times S) = \bar{B}_{r/2}(q)$. Since $D(B_r(q))$ is an open neighborhood of $\bar{B}_{r/2}(q)$ we must have $x_i \in D(B_r(q_0))$ for sufficiently large i in contradiction to the choice of the x_i.

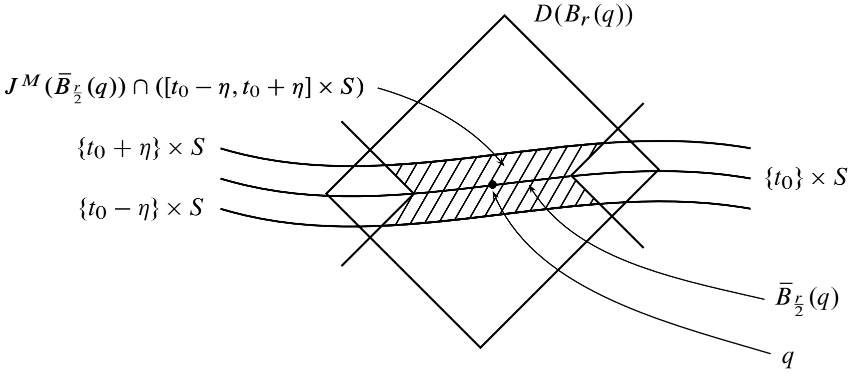


Figure 24. Definition of $\theta_r(q)$.

Lemma 3.2.9. *The function* $\theta_r : M \to (0, \infty]$ *is lower semi-continuous.*

Proof. Fix $q \in M$. Let $\varepsilon > 0$. We need to find a neighborhood U of q such that for all $q' \in U$ we have $\theta_r(q') \geq \theta_r(q) - \varepsilon$.

Put $\eta := \theta_r(q)$ and choose t_0 such that $q \in S_{t_0}$. Assume no such neighborhood U exists. Then there is a sequence $(q_i)_i$ in M such that $q_i \to q$ and $\theta_r(q_i) < \eta - \varepsilon$ for all i. All points to be considered will be contained in the compact set $([-T, T] \times S) \cap J^M(\overline{B}_r(q))$ for sufficiently big T and sufficiently large i. Let $q_i \in S_{t_i}$. Then $t_i \to t_0$ as $i \to \infty$.

Choose $x_i \in J^M(\overline{B}_{r/2}(q_i)) \cap ([t_i - \eta + \varepsilon, t_i + \eta - \varepsilon] \times S)$ but $x_i \notin D(B_r(q_i))$. This is possible because of $\theta_r(q_i) < \eta - \varepsilon$. Choose $y_i \in \overline{B}_{r/2}(q_i)$ such that $x_i \in J^M(y_i)$.

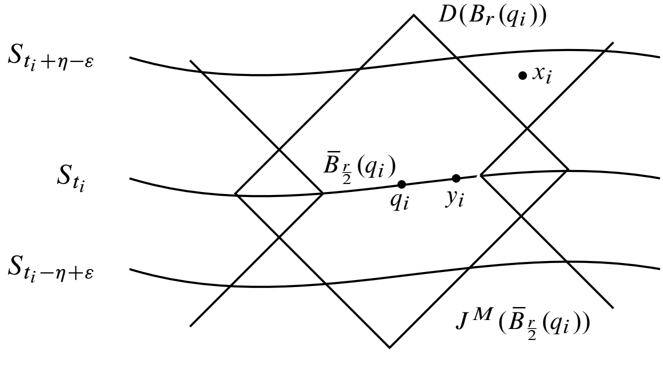


Figure 25. Construction of the sequence $(x_i)_i$.

After passing to a subsequence we may assume $x_i \to x$ and $y_i \to y$. From $q_i \to q$ and $y_i \in \overline{B}_{r/2}(q_i)$ we deduce $y \in \overline{B}_{r/2}(q)$. Since the causal relation "$\leq$" on a globally hyperbolic manifold is closed we conclude from $x_i \to x$, $y_i \to y$, and $x_i \in J^M(y_i)$ that $x \in J^M(y)$. Thus $x \in J^M(\overline{B}_{r/2}(q))$. Obviously, we also have $x \in [t_0 - \eta + \varepsilon, t_0 + \eta - \varepsilon] \times S$. From $\theta_r(q) = \eta > \eta - \varepsilon$ we conclude $x \in D(B_r(q))$.

Since $x_i \notin D(B_r(q_i))$ there is an inextendible causal curve c_i through x_i which does not intersect $B_r(q_i)$. Let z_i be the intersection of c_i with the Cauchy hypersurface S_{t_i}. After again passing to a subsequence we have $z_i \to z$ with $z \in S_{t_0}$. From $z_i \notin B_r(q_i)$ we conclude $z \notin B_r(q)$. Moreover, since c_i is causal we have $x_i \in J^M(z_i)$. The causal relation "$\leq$" is closed, hence $x \in J^M(z)$. Thus there exists an inextendible causal curve c through x and z. This curve does not meet $B_r(q)$ in contradiction to $x \in D(B_r(q))$. $\square$

Lemma 3.2.10. *For each compact subset $K \subset M$ there exists $\delta > 0$ such that for each $t \in \mathbb{R}$ and any $u_0, u_1 \in \mathcal{D}(S_t, E)$ with $\mathrm{supp}(u_j) \subset K$, $j = 1, 2$, there is a smooth solution u of $Pu = 0$ defined on $(t - \delta, t + \delta) \times S$ satisfying $u|_{S_t} = u_0$ and $\nabla_{\mathfrak{n}} u|_{S_t} = u_1$. Moreover, $\mathrm{supp}(u) \subset J^M(K \cap S_t)$.*

Proof. By Lemma 3.2.7 the function ρ admits a minimum on the compact set K. Hence there is a constant $r_0 > 0$ such that $\rho(q) > 2r_0$ for all $q \in K$. Choose $\delta > 0$ such that $\theta_{2r_0} > \delta$ on K. This is possible by Lemma 3.2.9.

Now fix $t \in \mathbb{R}$. Cover the compact set $S_t \cap K$ by finitely many balls $B_{r_0}(q_1), \dots, B_{r_0}(q_N)$, $q_j \in S_t \cap K$. Let $u_0, u_1 \in \mathcal{D}(S_t, E)$ with $\mathrm{supp}(u_j) \subset K$. Using a partition of unity write $u_0 = u_{0,1} + \dots + u_{0,N}$ with $\mathrm{supp}(u_{0,j}) \subset B_{r_0}(q_j)$ and similarly $u_1 = u_{1,1} + \dots + u_{1,N}$. The set $D(B_{2r_0}(q_j))$ is RCCSV. By Proposition 3.2.5 we can find a solution w_j of $Pw_j = 0$ on $D(B_{2r_0}(q_j))$ with $w_j|_{S_t} = u_{0,j}$ and $\nabla_{\mathfrak{n}} w_j|_{S_t} = u_{1,j}$. Moreover, $\mathrm{supp}(w_j) \subset J^M(B_{r_0}(q_j))$. From $J^M(B_{r_0}(q_j)) \cap (t - \delta, t + \delta) \times S \subset D(B_{2r_0}(q_j))$ we see that w_j is defined on $J^M(B_{r_0}(q_j)) \cap (t - \delta, t + \delta) \times S$. Extend w_j smoothly by zero to all of $(t - \delta, t + \delta) \times S$. Now $u := w_1 + \dots + w_N$ is a solution defined on $(t - \delta, t + \delta) \times S$ as required. $\square$

Now we are ready for the main theorem of this section.

Theorem 3.2.11. *Let M be a globally hyperbolic Lorentzian manifold and let $S \subset M$ be a spacelike Cauchy hypersurface. Let $\mathfrak{n}$ be the future directed timelike unit normal field along S. Let E be a vector bundle over M and let P be a normally hyperbolic operator acting on sections in E.*

Then for each $u_0, u_1 \in \mathcal{D}(S, E)$ and for each $f \in \mathcal{D}(M, E)$ there exists a unique $u \in C^\infty(M, E)$ satisfying $Pu = f$, $u|_S = u_0$, and $\nabla_{\mathfrak{n}} u|_S = u_1$.

Moreover, $\mathrm{supp}(u) \subset J^M(K)$ where $K = \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)$.

Proof. Uniqueness of the solution follows directly from Corollary 3.2.4. We have to show existence of a solution and the statement on its support.

Let $u_0, u_1 \in \mathcal{D}(S, E)$ and $f \in \mathcal{D}(M, E)$. Using a partition of unity $(\chi_j)_{j=1,\dots,m}$ we can write $u_0 = u_{0,1} + \dots + u_{0,m}$, $u_1 = u_{1,1} + \dots + u_{1,m}$ and $f = f_1 + \dots + f_m$ where $u_{0,j} = \chi_j u_0$, $u_{1,j} = \chi_j u_1$, and $f_j = \chi_j f$. We may assume that each χ_j (and hence each $u_{i,j}$ and f_j) have support in an open set as in Proposition 3.2.5. If we can solve the Cauchy problem on M for the data $(u_{0,j}, u_{1,j}, f_j)$, then we can add these solutions to obtain one for u_0, u_1, and f. Hence we can without loss of generality assume that there is an Ω as in Proposition 3.2.5 such that $K := \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f) \subset \Omega$.

By Theorem 1.3.13 the spacetime M is isometric to $\mathbb{R} \times S$ with a Lorentzian metric of the form $-\beta dt^2 + g_t$ where S corresponds to $\{0\} \times S$, and each $S_t := \{t\} \times S$ is a spacelike Cauchy hypersurface in M. Let u be the solution on Ω as asserted by Proposition 3.2.5. In particular, $\mathrm{supp}(u) \subset J^M(K)$. By choosing the partition of unity $(\chi_j)_j$ appropriately we can assume that K is so small that there exists an $\varepsilon > 0$ such that $((-\varepsilon, \varepsilon) \times S) \cap J^M(K) \subset \Omega$ and $K \subset (-\varepsilon, \varepsilon) \times S$.

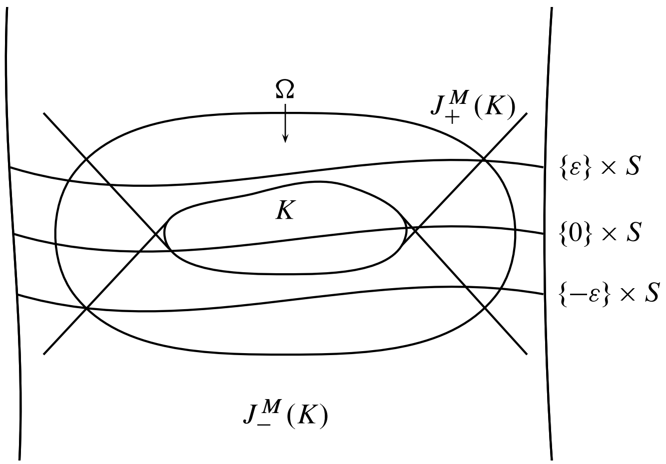


Figure 26. Construction of Ω and ε.

Hence we can extend u by 0 to a smooth solution on all of $(-\varepsilon, \varepsilon) \times S$. Now let T_+ be the supremum of all T for which u can be extended to a smooth solution on $(-\varepsilon, T) \times S$ with support contained in $J^M(K)$. On $[\varepsilon, T) \times S$ the equation to be solved is simply $Pu = 0$ because $\mathrm{supp}(f) \subset K$. If we have two extensions u and $\tilde{u}$ for $T < \tilde{T}$, then the restriction of $\tilde{u}$ to $(-\varepsilon, T) \times S$ must coincide with u by uniqueness. Note here that Corollary 3.2.4 applies because $(-\varepsilon, T) \times S$ is a globally hyperbolic manifold in its own right. Thus if we show $T_+ = \infty$ we obtain a solution on $(-\varepsilon, \infty) \times S$. Similarly considering the corresponding infimum T_- then yields a solution on all of $M = \mathbb{R} \times S$.

Assume that $T_+ < +\infty$. Put $\hat{K} := ([-\varepsilon, T_+] \times S) \cap J^M(K)$. By Lemma A.5.4 $\hat{K}$ is compact. Apply Lemma 3.2.10 to $\hat{K}$ and get $\delta > 0$ as in the Lemma. Fix $t < T_+$ such that $T_+ - t < \delta$ and still $K \subset (-\varepsilon, t) \times S$.

On $(t-\delta, t+\delta)\times S$ solve $Pw = 0$ with $w|_{S_t} = u|_{S_t}$ and $\nabla_{\mathfrak{n}} w|_{S_t} = \nabla_{\mathfrak{n}} u|_{S_t}$. This is possible by Lemma 3.2.10. On $(t-\eta, t+\delta)\times S$ the section f vanishes with $\eta > 0$ small enough. Thus w coincides with u on $(t-\eta, t)\times S$. Here again, Corollary 3.2.4 applies because $(t-\eta, t+\delta)\times S$ is a globally hyperbolic manifold in its own right. Hence w extends the solution u smoothly to $(-\varepsilon, t+\delta)\times S$. The support of this extension is still contained in $J^M(K)$ because

$$\begin{aligned}\operatorname{supp}\left(w|_{[t,t+\delta)\times S}\right) &\subset J_+^M\left(\operatorname{supp}(u|_{S_t})\cup\operatorname{supp}(\nabla_{\mathfrak{n}} u|_{S_t})\right)\\ &\subset J_+^M(\hat{K}\cap S_t)\subset J_+^M(J_+^M(K)) = J_+^M(K).\end{aligned}$$

Since $T_+ < t+\delta$ this contradicts the maximality of T_+. Therefore $T_+ = +\infty$. Similarly, one sees $T_- = -\infty$ which concludes the proof. □

The solution to the Cauchy problem depends continuously on the data.

Theorem 3.2.12. *Let M be a globally hyperbolic Lorentzian manifold and let $S\subset M$ be a spacelike Cauchy hypersurface. Let $\mathfrak{n}$ be the future directed timelike unit normal field along S. Let E be vector bundle over M and let P be a normally hyperbolic operator acting on sections in E.*

Then the map $\mathcal{D}(M,E)\oplus\mathcal{D}(S,E)\oplus\mathcal{D}(S,E)\to C^\infty(M,E)$ sending (f,u_0,u_1) to the unique solution u of the Cauchy problem $Pu = f$, $u|_S = u|_0$, $\nabla_{\mathfrak{n}} u = u_1$ is linear continuous.

Proof. The map $\mathcal{P}\colon C^\infty(M,E)\to C^\infty(M,E)\oplus C^\infty(S,E)\oplus C^\infty(S,E)$, $u\mapsto (Pu, u|_S, \nabla_{\mathfrak{n}} u)$, is obviously linear and continuous. Fix a compact subset $K\subset M$. Write $\mathcal{D}_K(M,E) := \{f\in\mathcal{D}(M,E)\mid \operatorname{supp}(f)\subset K\}$, $\mathcal{D}_K(S,E) := \{v\in\mathcal{D}(S,E)\mid\operatorname{supp}(v)\subset K\cap S\}$, and $\mathcal{V}_K := \mathcal{P}^{-1}(\mathcal{D}_K(M,E)\oplus\mathcal{D}_K(S,E)\oplus\mathcal{D}_K(S,E))$. Since $\mathcal{D}_K(M,E)\oplus\mathcal{D}_K(S,E)\oplus\mathcal{D}_K(S,E)\subset C^\infty(M,E)\oplus C^\infty(S,E)\oplus C^\infty(S,E)$ is a closed subset so is $\mathcal{V}_K\subset C^\infty(M,E)$. Both $\mathcal{V}_K$ and $\mathcal{D}_K(M,E)\oplus\mathcal{D}_K(S,E)\oplus\mathcal{D}_K(S,E)$ are therefore Fréchet spaces and $\mathcal{P}\colon\mathcal{V}_K\to\mathcal{D}_K(M,E)\oplus\mathcal{D}_K(S,E)\oplus\mathcal{D}_K(S,E)$ is linear, continuous and bijective. By the open mapping theorem [Reed–Simon1980, Thm. V.6, p. 132] the inverse mapping $\mathcal{P}^{-1}\colon\mathcal{D}_K(M,E)\oplus\mathcal{D}_K(S,E)\oplus\mathcal{D}_K(S,E)\to\mathcal{V}_K\subset C^\infty(M,E)$ is continuous as well.

Thus if $(f_j, u_{0,j}, u_{1,j})\to(f,u_0,u_1)$ in $\mathcal{D}(M,E)\oplus\mathcal{D}(S,E)\oplus\mathcal{D}(S,E)$, then we can choose a compact subset $K\subset M$ such that $(f_j,u_{0,j},u_{1,j})\to(f,u_0,u_1)$ in $\mathcal{D}_K(M,E)\oplus\mathcal{D}_K(S,E)\oplus\mathcal{D}_K(S,E)$ and we conclude $\mathcal{P}^{-1}(f_j,u_{0,j},u_{1,j})\to\mathcal{P}^{-1}(f,u_0,u_1)$. □

3.3 Fundamental solutions on globally hyperbolic manifolds

Using the knowledge about the Cauchy problem which we obtained in the previous section it is now not hard to find global fundamental solutions on a globally hyperbolic manifold.

Theorem 3.3.1. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.*

Then for every $x \in M$ there is exactly one fundamental solution $F_+(x)$ for P at x with past compact support and exactly one fundamental solution $F_-(x)$ for P at x with future compact support. They satisfy

(1) $\mathrm{supp}(F_\pm(x)) \subset J_\pm^M(x)$,

(2) *for each $\varphi \in \mathcal{D}(M, E^*)$ the maps $x \mapsto F_\pm(x)[\varphi]$ are smooth sections in E^* satisfying the differential equation $P^*(F_\pm(\cdot)[\varphi]) = \varphi$.*

Proof. Uniqueness of the fundamental solutions is a consequence of Corollary 3.1.2.

To show existence fix a foliation of M by spacelike Cauchy hypersurfaces S_t, $t \in \mathbb{R}$ as in Theorem 1.3.10. Let $\mathfrak{n}$ be the future directed unit normal field along the leaves S_t, and let $\varphi \in \mathcal{D}(M, E^*)$. Choose t so large that $\mathrm{supp}(\varphi) \subset I_-^M(S_t)$. By Theorem 3.2.11 there exists a unique $\chi_\varphi \in C^\infty(M, E^*)$ such that $P^*\chi_\varphi = \varphi$ and $\chi_\varphi|_{S_t} = (\nabla_{\mathfrak{n}}\chi_\varphi)|_{S_t} = 0$.

We check that χ_φ does not depend on the choice of t. Let $t < t'$ be such that $\mathrm{supp}(\varphi) \subset I_-^M(S_t) \subset I_-^M(S_{t'})$. Let χ_φ and χ'_φ be the corresponding solutions. Choose $t_- < t$ so that still $\mathrm{supp}(\varphi) \subset I_-^M(S_{t_-})$. The open subset $\hat{M} := \bigcup_{\tau > t_-} S_\tau \subset M$ is a globally hyperbolic Lorentzian manifold itself. Now χ'_φ satisfies $P^*\chi'_\varphi = 0$ on $\hat{M}$ with vanishing Cauchy data on $S_{t'}$. By Corollary 3.2.4 $\chi'_\varphi = 0$ on $\hat{M}$. In particular, χ'_φ has vanishing Cauchy data on S_t as well. Thus $\chi_\varphi - \chi'_\varphi$ has vanishing Cauchy data on S_t and solves $P^*(\chi_\varphi - \chi'_\varphi) = 0$ on all of M. Again by Corollary 3.2.4 we conclude $\chi_\varphi - \chi'_\varphi = 0$ on M.

Fix $x \in M$. By Theorem 3.2.12 χ_φ depends continuously on φ. Since the evaluation map $C^\infty(M, E) \to E_x$ is continuous, the map $\mathcal{D}(M, E^*) \to E_x^*$, $\varphi \mapsto \chi_\varphi(x)$, is also continuous. Thus $F_+(x)[\varphi] := \chi_\varphi(x)$ defines a distribution. By definition $P^*(F_+(\cdot)[\varphi]) = P^*\chi_\varphi = \varphi$.

Now $P^*\chi_{P^*\varphi} = P^*\varphi$, hence $P^*(\chi_{P^*\varphi} - \varphi) = 0$. Since both $\chi_{P^*\varphi}$ and φ vanish along S_t we conclude from Corollary 3.2.4 $\chi_{P^*\varphi} = \varphi$. Thus

$$(PF_+(x))[\varphi] = F_+(x)[P^*\varphi] = \chi_{P^*\varphi}(x) = \varphi(x) = \delta_x[\varphi].$$

Hence $F_+(x)$ is a fundamental solution of P at x.

It remains to show $\mathrm{supp}(F_+(x)) \subset J_+^M(x)$. Let $y \in M \setminus J_+^M(x)$. We have to construct a neighborhood of y such that for each test section $\varphi \in \mathcal{D}(M, E^*)$ whose support is contained in this neighborhood we have $F_+(x)[\varphi] = \chi_\varphi(x) = 0$. Since M is globally hyperbolic $J_+^M(x)$ is closed and therefore $J_+^M(x) \cap J_-^M(y') = \emptyset$ for all y' sufficiently close to y. We choose $y' \in I_+^M(y)$ and $y'' \in I_-^M(y)$ so close that $J_+^M(x) \cap J_-^M(y') = \emptyset$ and $\left(J_+^M(y'') \cap \bigcup_{t \le t'} S_t\right) \cap J_+^M(x) = \emptyset$ where $t' \in \mathbb{R}$ is such that $y' \in S_{t'}$.

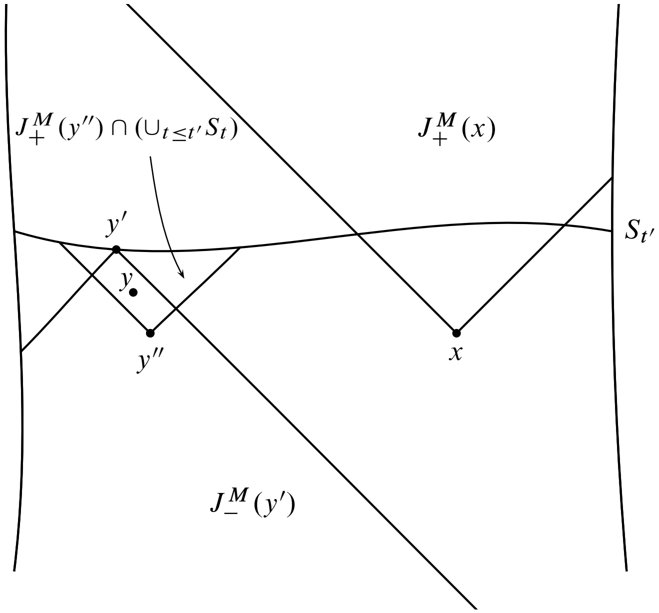


Figure 27. Global fundamental solution; construction of y, y' and y''.

Now $K := J_-^M(y') \cap J_+^M(y'')$ is a compact neighborhood of y. Let $\varphi \in \mathcal{D}(M, E^*)$ be such that $\mathrm{supp}(\varphi) \subset K$. By Theorem 3.2.11 $\mathrm{supp}(\chi_\varphi) \subset J_+^M(K) \cup J_-^M(K) \subset J_+^M(y'') \cup J_-^M(y')$. By the independence of χ_φ of the choice of $t > t'$ we have that χ_φ vanishes on $\bigcup_{t>t'} S_t$. Hence $\mathrm{supp}(\chi_\varphi) \subset \big(J_+^M(y'') \cap \bigcup_{t \le t'} S_t\big) \cup J_-^M(y')$ and is therefore disjoint from $J_+^M(x)$. Thus $F_+(x)[\varphi] = \chi_\varphi(x) = 0$ as required. □

3.4 Green's operators

Now we want to find "solution operators" for a given normally hyperbolic operator P. More precisely, we want to find operators which are inverses of P when restricted to suitable spaces of sections. We will see that existence of such operators is basically equivalent to the existence of fundamental solutions.

Definition 3.4.1. Let M be a time-oriented connected Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. A linear map $G_+ : \mathcal{D}(M, E) \to C^\infty(M, E)$ satisfying

(i) $P \circ G_+ = \mathrm{id}_{\mathcal{D}(M,E)}$,

(ii) $G_+ \circ P|_{\mathcal{D}(M,E)} = \mathrm{id}_{\mathcal{D}(M,E)}$,

(iii) $\mathrm{supp}(G_+\varphi) \subset J_+^M(\mathrm{supp}(\varphi))$ for all $\varphi \in \mathcal{D}(M, E)$, is called an *advanced Green's operator for* P. Similarly, a linear map $G_-\colon \mathcal{D}(M,E) \to C^\infty(M,E)$ satisfying (i), (ii), and

(iii′) $\mathrm{supp}(G_-\varphi) \subset J_-^M(\mathrm{supp}(\varphi))$ for all $\varphi \in \mathcal{D}(M,E)$

instead of (iii) is called a *retarded Green's operator for* P.

Fundamental solutions and Green's operators are closely related.

Proposition 3.4.2. *Let M be a time-oriented connected Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.*

If $F_\pm(x)$ is a family of advanced or retarded fundamental solutions for the adjoint operator P^* and if $F_\pm(x)$ depend smoothly on x in the sense that $x \mapsto F_\pm(x)[\varphi]$ is smooth for each test section φ and satisfies the differential equation $P(F_\pm(\cdot)[\varphi]) = \varphi$, then

$$(G_\pm\varphi)(x) := F_\mp(x)[\varphi] \tag{3.8}$$

defines advanced or retarded Green's operators for P respectively. Conversely, given Green's operators $G_\pm$ for P, then (3.8) defines fundamental solutions for P^* depending smoothly on x and satisfying $P(F_\pm(\cdot)[\varphi]) = \varphi$ for each test section φ.

Proof. Let $F_\pm(x)$ be a family of advanced and retarded fundamental solutions for the adjoint operator P^* respectively. Let $F_\pm(x)$ depend smoothly on x and suppose the differential equation $P(F_\pm(\cdot)[\varphi]) = \varphi$ holds. By definition we have

$$P(G_\pm\varphi) = P(F_\mp(\cdot)[\varphi]) = \varphi$$

thus showing (i). Assertion (ii) follows from the fact that the $F_\pm(x)$ are fundamental solutions,

$$G_\pm(P\varphi)(x) = F_\mp(x)[P\varphi] = P^*F_\mp(x)[\varphi] = \delta_x[\varphi] = \varphi(x).$$

To show (iii) let $x \in M$ such that $(G_+\varphi)(x) \neq 0$. Since $\mathrm{supp}(F_-(x)) \subset J_-^M(x)$ the support of φ must hit $J_-^M(x)$. Hence $x \in J_+^M(\mathrm{supp}(\varphi))$ and therefore $\mathrm{supp}(G_+\varphi) \subset J_+^M(\mathrm{supp}(\varphi))$. The argument for G_- is analogous.

The converse is similar. □

Theorem 3.3.1 immediately yields

Corollary 3.4.3. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.*

Then there exist unique advanced and retarded Green's operators $G_\pm\colon \mathcal{D}(M,E) \to C^\infty(M,E)$ for P. □

Lemma 3.4.4. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let $G_\pm$ be the Green's operators for P and $G_\pm^*$ the Green's operators for the adjoint operator P^*. Then*

$$\int_M (G_\pm^*\varphi)\cdot\psi \,\mathrm{dV} = \int_M \varphi\cdot(G_\mp\psi)\,\mathrm{dV} \tag{3.9}$$

holds for all $\varphi \in \mathcal{D}(M,E^*)$ and $\psi \in \mathcal{D}(M,E)$.

Proof. For the Green's operators we have $PG_\pm = \mathrm{id}_{\mathcal{D}(M,E)}$ and $P^* G^*_\pm = \mathrm{id}_{\mathcal{D}(M,E^*)}$ and hence

$$\begin{aligned}\int_M (G^*_\pm \varphi) \cdot \psi \,\mathrm{dV} &= \int_M (G^*_\pm \varphi) \cdot (P G_\mp \psi) \,\mathrm{dV} \\ &= \int_M (P^* G^*_\pm \varphi) \cdot (G_\mp \psi) \,\mathrm{dV} \\ &= \int_M \varphi \cdot (G_\mp \psi) \,\mathrm{dV}.\end{aligned}$$

Notice that $\mathrm{supp}(G_\pm \varphi) \cap \mathrm{supp}(G_\mp \psi) \subset J^M_\pm(\mathrm{supp}(\varphi)) \cap J^M_\mp(\mathrm{supp}(\psi))$ is compact in a globally hyperbolic manifold so that the partial integration in the second equation is justified. □

Notation 3.4.5. We write $C^\infty_{\mathrm{sc}}(M,E)$ for the set of all $\varphi \in C^\infty(M,E)$ for which there exists a compact subset $K \subset M$ such that $\mathrm{supp}(\varphi) \subset J^M(K)$. Obviously, $C^\infty_{\mathrm{sc}}(M,E)$ is a vector subspace of $C^\infty(M,E)$.

The subscript "sc" should remind the reader of "spacelike compact". Namely, if M is globally hyperbolic and $\varphi \in C^\infty_{\mathrm{sc}}(M,E)$, then for every Cauchy hypersurface $S \subset M$ the support of $\varphi|_S$ is contained in $S \cap J^M(K)$ hence compact by Corollary A.5.4. In this sense sections in $C^\infty_{\mathrm{sc}}(M,E)$ have spacelike compact support.

Definition 3.4.6. We say a sequence of elements $\varphi_j \in C^\infty_{\mathrm{sc}}(M,E)$ *converges in* $C^\infty_{\mathrm{sc}}(M,E)$ *to* $\varphi \in C^\infty_{\mathrm{sc}}(M,E)$ if there exists a compact subset $K \subset M$ such that

$$\mathrm{supp}(\varphi_j), \mathrm{supp}(\varphi) \subset J^M(K)$$

for all j and

$$\|\varphi_j - \varphi\|_{C^k(K',E)} \to 0$$

for all $k \in \mathbb{N}$ and all compact subsets $K' \subset M$.

If G_+ and G_- are advanced and retarded Green's operators for P respectively, then we get a linear map

$$G := G_+ - G_- \colon \mathcal{D}(M,E) \to C^\infty_{\mathrm{sc}}(M,E).$$

Theorem 3.4.7. *Let M be a connected time-oriented Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let G_+ and G_- be advanced and retarded Green's operators for P respectively.*

Then the sequence of linear maps

$$0 \to \mathcal{D}(M,E) \xrightarrow{P} \mathcal{D}(M,E) \xrightarrow{G} C^\infty_{\mathrm{sc}}(M,E) \xrightarrow{P} C^\infty_{\mathrm{sc}}(M,E) \tag{3.10}$$

is a complex, i.e., the composition of any two subsequent maps is zero. The complex is exact at the first $\mathcal{D}(M,E)$. If M is globally hyperbolic, then the complex is exact everywhere.

Proof. Properties (i) and (ii) in Definition 3.4.1 of Green's operators directly yield $G \circ P = 0$ and $P \circ G = 0$, both on $\mathcal{D}(M, E)$. Properties (iii) and (iii′) ensure that G maps $\mathcal{D}(M, E)$ to $C^\infty_{\mathrm{sc}}(M, E)$. Hence the sequence of linear maps forms a complex.

Exactness at the first $\mathcal{D}(M, E)$ means that

$$P : \mathcal{D}(M, E) \to \mathcal{D}(M, E)$$

is injective. To see injectivity let $\varphi \in \mathcal{D}(M, E)$ with $P\varphi = 0$. Then $\varphi = G_+ P\varphi = G_+ 0 = 0$.

From now on let M be globally hyperbolic. Let $\varphi \in \mathcal{D}(M, E)$ with $G\varphi = 0$, i.e., $G_+\varphi = G_-\varphi$. We put $\psi := G_+\varphi = G_-\varphi \in C^\infty(M, E)$ and we see $\mathrm{supp}(\psi) = \mathrm{supp}(G_+\varphi) \cap \mathrm{supp}(G_-\varphi) \subset J^M_+(\mathrm{supp}(\varphi)) \cap J^M_-(\mathrm{supp}(\varphi))$. Since (M, g) is globally hyperbolic $J^M_+(\mathrm{supp}(\varphi)) \cap J^M_-(\mathrm{supp}(\varphi))$ is compact, hence $\psi \in \mathcal{D}(M, E)$. From $P(\psi) = P(G_+(\varphi)) = \varphi$ we see that $\varphi \in P(\mathcal{D}(M, E))$. This shows exactness at the second $\mathcal{D}(M, E)$.

Finally, let $\varphi \in C^\infty_{\mathrm{sc}}(M, E)$ such that $P\varphi = 0$. Without loss of generality we may assume that $\mathrm{supp}(\varphi) \subset I^M_+(K) \cup I^M_-(K)$ for a compact subset K of M. Using a partition of unity subordinated to the open covering $\{I^M_+(K), I^M_-(K)\}$ write φ as $\varphi = \varphi_1 + \varphi_2$ where $\mathrm{supp}(\varphi_1) \subset I^M_-(K) \subset J^M_-(K)$ and $\mathrm{supp}(\varphi_2) \subset I^M_+(K) \subset J^M_+(K)$. For $\psi := -P\varphi_1 = P\varphi_2$ we see that $\mathrm{supp}(\psi) \subset J^M_-(K) \cap J^M_+(K)$, hence $\psi \in \mathcal{D}(M, E)$.

We check that $G_+\psi = \varphi_2$. For all $\chi \in \mathcal{D}(M, E^*)$ we have

$$\int_M \chi \cdot (G_+ P\varphi_2)\,\mathrm{dV} = \int_M (G^*_-\chi) \cdot (P\varphi_2)\,\mathrm{dV} = \int_M (P^* G^*_-\chi) \cdot \varphi_2\,\mathrm{dV} = \int_M \chi \cdot \varphi_2\,\mathrm{dV}$$

where G^*_- is the Green's operator for the adjoint operator P^* according to Lemma 3.4.4. Notice that for the second equation we use the fact that $\mathrm{supp}(\varphi_2) \cap \mathrm{supp}(G^*_-\chi) \subset J^M_+(K) \cap J^M_-(\mathrm{supp}(\chi))$ is compact. Similarly, one shows $G_-\psi = -\varphi_1$.

Now $G\psi = G_+\psi - G_-\psi = \varphi_2 + \varphi_1 = \varphi$, hence φ is in the image of G. □

Proposition 3.4.8. *Let M be a globally hyperbolic Lorentzian manifold, let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let G_+ and G_- be the advanced and retarded Green's operators for P respectively.*

Then $G_\pm : \mathcal{D}(M, E) \to C^\infty_{\mathrm{sc}}(M, E)$ and all maps in the complex (3.10) are sequentially continuous.

Proof. The maps $P : \mathcal{D}(M, E) \to \mathcal{D}(M, E)$ and $P : C^\infty_{\mathrm{sc}}(M, E) \to C^\infty_{\mathrm{sc}}(M, E)$ are sequentially continuous simply because P is a differential operator. It remains to show that $G : \mathcal{D}(M, E) \to C^\infty_{\mathrm{sc}}(M, E)$ is sequentially continuous.

Let $\varphi_j, \varphi \in \mathcal{D}(M, E)$ and $\varphi_j \to \varphi$ in $\mathcal{D}(M, E)$ for all j. Then there exists a compact subset $K \subset M$ such that $\mathrm{supp}(\varphi_j), \mathrm{supp}(\varphi) \subset K$. Hence $\mathrm{supp}(G\varphi_j), \mathrm{supp}(G\varphi) \subset J^M(K)$ for all j. From the proof of Theorem 3.3.1 we know that $G_+\varphi$ coincides with the solution u to the Cauchy problem $Pu = \varphi$ with initial conditions $u|_{S_-} = (\nabla_{\mathfrak{n}} u)|_{S_-} = 0$ where $S_- \subset M$ is a spacelike Cauchy hypersurface such that

$K \subset I_+^M(S_-)$. Theorem 3.2.12 tells us that if $\varphi_j \to \varphi$ in $\mathcal{D}(M, E)$, then the solutions $G_+\varphi_j \to G_+\varphi$ in $C^\infty(M, E)$. The proof for G_- is analogous and the statement for G follows. ☐

Remark 3.4.9. Green's operators need not exist for any normally hyperbolic operator on any spacetime. For example consider a compact spacetime M and the d'Alembert operator acting on real functions. Note that in this case $\mathcal{D}(M, \mathbb{R}) = C^\infty(M)$. If there existed Green's operators the d'Alembert operator would be injective. But any constant function belongs to the kernel of the operator.

3.5 Non-globally hyperbolic manifolds

Globally hyperbolic Lorentzian manifolds turned out to form a good class for the solution theory of normally hyperbolic operators. We have unique advanced and retarded fundamental solutions and Green's operators. The Cauchy problem is well-posed. Some of these analytical features survive when we pass to more general Lorentzian manifolds. We will see that we still have existence (but not uniqueness) of fundamental solutions and Green's operators if the manifold can be embedded in a suitable way as an open subset into a globally hyperbolic manifold such that the operator extends. Moreover, we will see that conformal changes of the Lorentzian metric do not alter the basic analytical properties. To illustrate this we construct Green's operators for the Yamabe operator on the important anti-deSitter spacetime which is not globally hyperbolic.

Proposition 3.5.1. *Let M be a time-oriented connected Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let $G_\pm$ be Green's operators for P. Let $\Omega \subset M$ be a causally compatible connected open subset.*

Define $\tilde{G}_\pm : \mathcal{D}(\Omega, E) \to C^\infty(\Omega, E)$ by

$$\tilde{G}_\pm(\varphi) := G_\pm(\varphi_{\mathrm{ext}})|_\Omega.$$

Here $\mathcal{D}(\Omega, E) \to \mathcal{D}(M, E)$, $\varphi \mapsto \varphi_{\mathrm{ext}}$, denotes extension by zero.

Then $\tilde{G}_+$ and $\tilde{G}_-$ are advanced and retarded Green's operators for the restriction of P to Ω respectively.

Proof. Denote the restriction of P to Ω by $\tilde{P}$. To show (i) in Definition 3.4.1 we check for $\varphi \in \mathcal{D}(\Omega, E)$

$$\tilde{P}\tilde{G}_\pm\varphi = \tilde{P}(G_\pm(\varphi_{\mathrm{ext}})|_\Omega) = P(G_\pm(\varphi_{\mathrm{ext}}))|_\Omega = \varphi_{\mathrm{ext}}|_\Omega = \varphi.$$

Similarly, we see for (ii)

$$\tilde{G}_\pm\tilde{P}\varphi = G_\pm((\tilde{P}\varphi)_{\mathrm{ext}})|_\Omega = G_\pm(P\varphi_{\mathrm{ext}})|_\Omega = \varphi_{\mathrm{ext}}|_\Omega = \varphi.$$

For (iii) we need that Ω is a causally compatible subset of M.

$$\begin{aligned}\operatorname{supp}(\tilde{G}_\pm\varphi) &= \operatorname{supp}(G_\pm(\varphi_{\mathrm{ext}})|_\Omega)\\ &= \operatorname{supp}(G_\pm(\varphi_{\mathrm{ext}}))\cap\Omega\\ &\subset J_\pm^M(\operatorname{supp}(\varphi_{\mathrm{ext}}))\cap\Omega\\ &= J_\pm^M(\operatorname{supp}(\varphi))\cap\Omega\\ &= J_\pm^\Omega(\operatorname{supp}(\varphi)).\end{aligned}$$ □

Example 3.5.2. In Minkowski space every convex open subset Ω is causally compatible. Proposition 3.5.1 shows the existence of an advanced and a retarded Green's operator for any normally hyperbolic operator on Ω which extends to a normally hyperbolic operator on M.

On the other hand, we have already noticed in Remark 3.1.5 that on convex domains the advanced and retarded fundamental solutions need not be unique. Thus the Green's operators $G_\pm$ are not unique in general.

The proposition fails if we drop the condition on Ω to be a causally compatible subset of M.

Example 3.5.3. For non-convex domains Ω in Minkowski space $M = \mathbb{R}^n$ causal compatibility does not hold in general, see Figure 7 on page 19. For any $\varphi \in \mathcal{D}(\Omega, E)$ the proof of Proposition 3.5.1 shows that $\operatorname{supp}(\tilde{G}_\pm\varphi) \subset J_\pm^M(\operatorname{supp}(\varphi)) \cap \Omega$. Now, if $J_\pm^\Omega(p)$ is a proper subset of $J_\pm^M(p) \cap \Omega$ there is no reason why $\operatorname{supp}(\tilde{G}_\pm\varphi)$ should be a subset of $J_\pm^\Omega(\operatorname{supp}(\varphi))$. Hence $\tilde{G}_\pm$ are not Green's operators in general.

Example 3.5.4. We consider the *Einstein cylinder* $M = \mathbb{R} \times S^{n-1}$ equipped with the product metric $g = -dt^2 + \mathrm{can}_{S^{n-1}}$ where $\mathrm{can}_{S^{n-1}}$ denotes the canonical Riemannian metric of constant sectional curvature 1 on the sphere. Since S^{n-1} is compact, the Einstein cylinder is globally hyperbolic, compare Example 1.3.11.

We put $\Omega := \mathbb{R} \times S_+^{n-1}$ where $S_+^{n-1} := \{(z_1, \dots, z_n) \in S^{n-1} \mid z_n > 0\}$ denotes the northern hemisphere. Let p and q be two points in Ω which can be joined by a causal curve $c\colon [0,1] \to M$ in M. We write $c(s) = (t(s), x(s))$ with $x(s) \in S^{n-1}$. After reparametrization we may assume that the curve x in S^{n-1} is parametrized proportionally to arclength, $\mathrm{can}_{S^{n-1}}(x', x') \equiv \xi$ where ξ is a nonnegative constant.

Since S_+^{n-1} is a geodesically convex subset of the Riemannian manifold S^{n-1} there is a curve $y\colon [0,1] \to S_+^{n-1}$ with the same end points as x and of length at most the length of x. If we parametrize y proportionally to arclength this means $\mathrm{can}_{S^{n-1}}(y', y') \equiv \eta \le \xi$. The curve c being causal means $0 \ge g(c', c') = -(t')^2 + \mathrm{can}_{S^{n-1}}(x', x')$, i.e.,

$$(t')^2 \ge \xi.$$

This implies $(t')^2 \ge \eta$ which in turn is equivalent to the curve $\tilde{c} := (t, y)$ being causal. Thus p and q can be joined by a causal curve which stays in Ω. Therefore Ω is a causally compatible subset of the Einstein cylinder.

Next we study conformal changes of the metric. Let M be a time-oriented connected Lorentzian manifold. Denote the Lorentzian metric by g. Let $f : M \to \mathbb{R}$ be a positive smooth function. Denote the conformally related metric by $\tilde{g} := f \cdot g$. This means that $\tilde{g}(X, Y) = f(p) \cdot g(X, Y)$ for all $X, Y \in T_pM$. The causal type of tangent vectors and curves is unaffected by this change of metric. Therefore all causal concepts such as the chronological or causal future and past remain unaltered by a conformal change of the metric. Similarly, the causality conditions are unaffected. Hence (M, g) is globally hyperbolic if and only if $(M, \tilde{g})$ is globally hyperbolic.

Let us denote by g^* and $\tilde{g}^*$ the metrics on the cotangent bundle T^*M induced by g and $\tilde{g}$ respectively. Then we have $\tilde{g}^* = \frac{1}{f} g^*$.

Let $\tilde{P}$ be a normally hyperbolic operator with respect to $\tilde{g}$. Put $P := f \cdot \tilde{P}$, more precisely,

$$P(\varphi) = f \cdot \tilde{P}(\varphi) \tag{3.11}$$

for all φ. Since the principal symbol of $\tilde{P}$ is given by $\tilde{g}^*$, the principal symbol of P is given by g^*,

$$\sigma_P(\xi) = f \cdot \sigma_{\tilde{P}}(\xi) = -f \cdot \tilde{g}^*(\xi, \xi) \cdot \mathrm{id} = -g^*(\xi, \xi) \cdot \mathrm{id}.$$

Thus P is normally hyperbolic for g. Now suppose we have an advanced or a retarded Green's operator G_+ or G_- for P. We define $\tilde{G}_\pm : \mathcal{D}(M, E) \to C^\infty(M, E)$ by

$$\tilde{G}_\pm \varphi := G_\pm (f \cdot \varphi) . \tag{3.12}$$

We see that

$$\tilde{G}_\pm(\tilde{P}\varphi) = G_\pm(f \cdot \tfrac{1}{f} \cdot P\varphi) = G_\pm(P\varphi) = \varphi$$

and

$$\tilde{P}(\tilde{G}_\pm\varphi) = \tfrac{1}{f} \cdot P(G_\pm(f \cdot \varphi)) = \tfrac{1}{f} \cdot f \cdot \varphi = \varphi.$$

Multiplication by a nowhere vanishing function does not change supports, hence

$$\mathrm{supp}(\tilde{G}_\pm\varphi) = \mathrm{supp}(G_\pm(f\,\varphi)) \subset J^M_\pm(\mathrm{supp}(f\,\varphi)) = J^M_\pm(\mathrm{supp}(\varphi)).$$

Notice again that $J^M_\pm$ is the same for g and for $\tilde{g}$. We have thus shown that $\tilde{G}_\pm$ is a Green's operator for $\tilde{P}$. We summarize:

Proposition 3.5.5. *Let M be a time-oriented connected Lorentzian manifold with Lorentzian metric g. Let $f : M \to \mathbb{R}$ be a positive smooth function and denote the conformally related metric by $\tilde{g} := f \cdot g$.*

Then (3.11) yields a 1-1-correspondence $P \leftrightarrow \tilde{P}$ between normally hyperbolic operators for g and such operators for $\tilde{g}$. Similarly, (3.12) yields a 1-1-correspondence $G_\pm \leftrightarrow \tilde{G}_\pm$ for their Green's operators. □

This discussion can be slightly generalized.

Remark 3.5.6. Let (M, g) be a time-oriented connected Lorentzian manifold. Let P be a normally hyperbolic operator on M for which advanced and retarded Green's operators G_+ and G_- exist. Let $f_1, f_2 \colon M \to \mathbb{R}$ be positive smooth functions. Then the operator $\tilde{P} := \frac{1}{f_1} \cdot P \cdot \frac{1}{f_2}$, given by

$$\tilde{P}(\varphi) = \tfrac{1}{f_1} \cdot P\big(\tfrac{1}{f_2} \cdot \varphi\big) \tag{3.13}$$

for all φ, possesses advanced and retarded Green's operators $\tilde{G}_\pm$. They can be defined in analogy to (3.12):

$$\tilde{G}_\pm(\varphi) := f_2 \cdot G_\pm(f_1 \cdot \varphi).$$

As above one gets $\tilde{P}\tilde{G}_\pm(\varphi) = \varphi$ and $\tilde{G}_\pm(\tilde{P}\varphi) = \varphi$ for all $\varphi \in \mathcal{D}(M, E)$. Operators $\tilde{P}$ of the form (3.13) are normally hyperbolic with respect to the conformally related metric $\tilde{g} = f_1 \cdot f_2 \cdot g$.

Combining Propositions 3.5.1 and 3.5.5 we get:

Corollary 3.5.7. *Let $(\tilde{M}, \tilde{g})$ be a time-oriented connected Lorentzian manifold which can be conformally embedded as a causally compatible open subset Ω into the globally hyperbolic manifold (M, g). Hence on Ω we have $\tilde{g} = f \cdot g$ for some positive function $f \in C^\infty(\Omega, \mathbb{R})$.*

Let $\tilde{P}$ be a normally hyperbolic operator on $(\tilde{M}, \tilde{g})$ and let P be the operator on Ω defined as in (3.11). Assume that P can be extended to a normally hyperbolic operator on the whole manifold (M, g). Then the operator $\tilde{P}$ possesses advanced and retarded Green's operators. Uniqueness is lost in general. □

In the remainder of this section we will show that the preceding considerations can be applied to an important example in general relativity: *anti-deSitter spacetime*. We will show that it can be conformally embedded into the Einstein cylinder. The image of this embedding is the set Ω in Example 3.5.4. Hence we realize anti-deSitter spacetime conformally as a causally compatible subset of a globally hyperbolic Lorentzian manifold.

For an integer $n \geq 2$, one defines the n-dimensional *pseudohyperbolic space*

$$H_1^n := \{x \in \mathbb{R}^{n+1} \mid \langle\langle x, x \rangle\rangle = -1\},$$

where $\langle\langle x, y \rangle\rangle := -x_0 y_0 - x_1 y_1 + \sum_{j=2}^n x_j y_j$ for all $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$ in $\mathbb{R}^{n+1}$. With the induced metric (also denoted by $\langle\langle \cdot, \cdot \rangle\rangle$) H_1^n becomes a connected Lorentzian manifold with constant sectional curvature -1, see e.g. [O'Neill1983, Chap. 4, Prop. 29].

Lemma 3.5.8. *There exists a conformal diffeomorphism*

$$\Psi \colon \big(S^1 \times S_+^{n-1}, -\mathrm{can}_{S^1} + \mathrm{can}_{S_+^{n-1}}\big) \to \big(H_1^n, \langle\langle \cdot, \cdot \rangle\rangle\big)$$

such that for any $(p, x) \in S^1 \times S_+^{n-1} \subset S^1 \times \mathbb{R}^n$ one has

$$\big(\psi^* \langle\langle \cdot, \cdot \rangle\rangle\big)_{(p,x)} = \frac{1}{x_n^2}\big(-\mathrm{can}_{S^1} + \mathrm{can}_{S_+^{n-1}}\big).$$

Proof. We first construct an isometry between the pseudohyperbolic space and

$$\left(S^1 \times H^{n-1}, -y_1^2 \operatorname{can}_{S^1} + \operatorname{can}_{H^{n-1}}\right),$$

where $H^{n-1} := \{(y_1, \dots, y_n) \in \mathbb{R}^n \mid y_1 > 0 \text{ and } -y_1^2 + \sum_{j=2}^n y_j^2 = -1\}$ is the $(n-1)$-dimensional hyperbolic space. The hyperbolic metric $\operatorname{can}_{H^{n-1}}$ is induced by the Minkowski metric on $\mathbb{R}^n$. Then $(H^{n-1}, \operatorname{can}_{H^{n-1}})$ is a Riemannian manifold with constant sectional curvature -1. Define the map

$$\begin{aligned} \Phi\colon S^1 \times H^{n-1} &\to H_1^n, \\ (p = (p_0, p_1), y = (y_1, \dots, y_n)) &\mapsto (y_1 p_0, y_1 p_1, y_2, \dots, y_n) \in \mathbb{R}^{n+1}. \end{aligned}$$

This map is clearly well defined because $-y_1^2 \underbrace{(p_0^2 + p_1^2)}_{=1} + y_2^2 + \cdots y_n^2 = -y_1^2 + \sum_{j=2}^n y_j^2 = -1$. The inverse map is given by

$$\Phi^{-1}(x) = \left(\left(\frac{x_0}{\sqrt{x_0^2 + x_1^2}}, \frac{x_1}{\sqrt{x_0^2 + x_1^2}}\right), \left(\sqrt{x_0^2 + x_1^2}, x_2, \dots, x_n\right)\right).$$

Geometrically, the map Φ can be interpreted as follows: For any point $p = (p_0, p_1) \in S^1$, consider the hyperplane $\mathcal{H}_p$ of $\mathbb{R}^{n+1}$ defined by

$$\mathcal{H}_p := \mathbb{R} \cdot (p_0, p_1, 0, \dots, 0) \oplus \mathbb{R}^{n-1},$$

where the point $(p_0, p_1, 0, \dots, 0)$ lies in $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n-1}$ is identified with the subspace $\{(0, 0, w_2, \dots, w_n) \mid w_j \in \mathbb{R}\} \subset \mathbb{R}^{n+1}$. If $\{e_2, \dots, e_n\}$ is the canonical basis of this $\mathbb{R}^{n-1}$, then

$$\mathcal{B}_p := \{e_1 := (p_0, p_1, 0 \dots, 0), e_2, \dots, e_n\}$$

is a Lorentz orthonormal basis of $\mathcal{H}_p$ with respect to the metric induced by $\langle\langle \cdot , \cdot \rangle\rangle$. Define $H^{n-1}(p)$ as the hyperbolic space of $(\mathcal{H}_p, \langle\langle \cdot , \cdot \rangle\rangle)$ in this basis. More precisely, $H^{n-1}(p) = \left\{\sum_{j=1}^n \eta_j e_j \in \mathcal{H}_p \mid \eta_1 > 0, -\eta_1^2 + \sum_{j=2}^n \eta_j^2 = -1\right\}$. Then $y \mapsto \Phi(p, y)$ yields an isometry from Minkowski space to $\mathcal{H}_p$ which restricts to an isometry $H^{n-1} \to H^{n-1}(p)$.

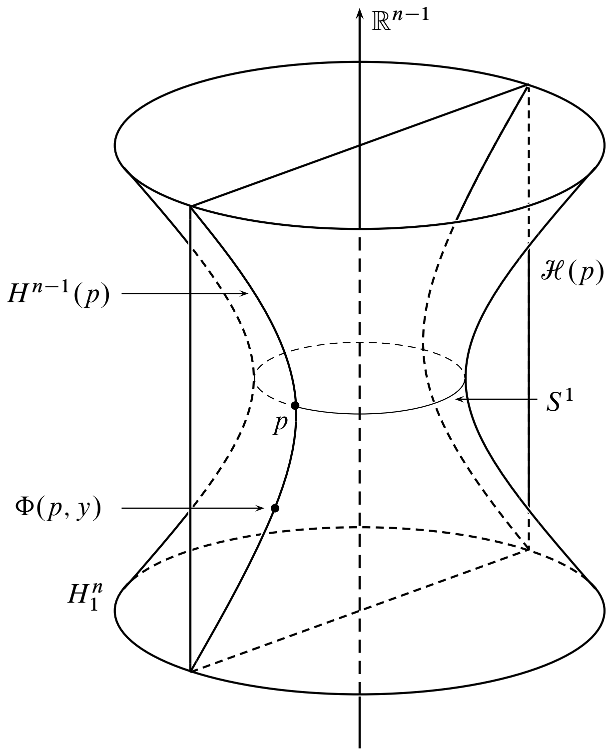


Figure 28. Pseudohyperbolic space; construction of $\Phi(p, y)$.

Let now $(p, y) \in S^1 \times H^{n-1}$ and $X = (X^1, X^{n-1}) \in T_pS^1 \oplus T_yH^{n-1}$. Then the differential of Φ at (p, y) is given by

$$d_{(p,y)}\Phi(X) = \left(y_1 X^1 + X_1^{n-1} p, X_2^{n-1}, \ldots, X_n^{n-1}\right).$$

Therefore the pull-back of the metric on H_1^n via Φ can be computed to yield

$$\begin{aligned}
(\Phi^* \langle\!\langle \cdot, \cdot \rangle\!\rangle)_{(p,y)}(X, X) &= \langle\!\langle d_{(p,y)}\Phi(X), d_{(p,y)}\Phi(X) \rangle\!\rangle \\
&= y_1^2 \langle\!\langle X^1, X^1 \rangle\!\rangle + (X_1^{n-1})^2 \langle\!\langle p, p \rangle\!\rangle + \sum_{j=2}^{n} (X_j^{n-1})^2 \\
&= -y_1^2 \{(X_0^1)^2 + (X_1^1)^2\} - (X_1^{n-1})^2 + \sum_{j=2}^{n} (X_j^{n-1})^2 \\
&= -y_1^2 \operatorname{can}_{S^1}(X^1, X^1) + \operatorname{can}_{H^{n-1}}(X^{n-1}, X^{n-1}).
\end{aligned}$$

Hence Φ is an isometry

$$\left(S^1 \times H^{n-1}, -y_1^2 \operatorname{can}_{S^1} + \operatorname{can}_{H^{n-1}}\right) \to \left(H_1^n, \langle\!\langle \cdot, \cdot \rangle\!\rangle\right).$$

The stereographic projection from the south pole

$$\begin{aligned}\pi\colon S^{n-1}_+ &\to H^{n-1},\\ x=(x_1,\dots,x_n)&\mapsto \frac{1}{x_n}(1,x_1,\dots,x_{n-1}),\end{aligned}$$

is a conformal diffeomorphism. It is easy to check that π is a well-defined diffeomorphism with inverse given by $y=(y_1,\dots,y_n)\mapsto \frac{1}{y_1}(y_2,\dots,y_n,1)$. For any $x\in S^{n-1}_+$ and $X\in T_xS^{n-1}_+$ the differential of π at x is given by

$$\begin{aligned}d_x\pi(X)&=\frac{1}{x_n}(0,X_1,\dots,X_{n-1})-\frac{X_n}{x_n^2}(1,x_1,\dots,x_{n-1})\\ &=\frac{1}{x_n^2}(-X_n,x_nX_1-x_1X_n,\dots,x_nX_{n-1}-x_{n-1}X_n).\end{aligned}$$

Therefore we get for the pull-back of the hyperbolic metric

$$\begin{aligned}(\pi^*\mathrm{can}_{H^{n-1}})_x(X,X)&=\frac{1}{x_n^4}\Big\{-X_n^2+\sum_{j=1}^{n-1}(x_nX_j-x_jX_n)^2\Big\}\\ &=\frac{1}{x_n^4}\Big\{-X_n^2+x_n^2\sum_{j=1}^{n-1}X_j^2-2x_nX_n\underbrace{\sum_{j=1}^{n-1}x_jX_j}_{=-x_nX_n}+X_n^2\underbrace{\sum_{j=1}^{n-1}x_j^2}_{=1-x_n^2}\Big\}\\ &=\frac{1}{x_n^2}\sum_{j=1}^{n-1}X_j^2+\frac{X_n^2}{x_n^2}\\ &=\frac{1}{x_n^2}\sum_{j=1}^{n}X_j^2,\end{aligned}$$

that is, $(\pi^*\mathrm{can}_{H^{n-1}})_x=\frac{1}{x_n^2}(\mathrm{can}_{S^{n-1}_+})_x$. We obtain an explicit diffeomorphism

$$\begin{aligned}\Psi:=\Phi\circ(\mathrm{id}\times\pi)\colon S^1\times S^{n-1}_+&\to H^n_1,\\ (p=(p_0,p_1),x=(x_1,\dots,x_n))&\mapsto\frac{1}{x_n}(p_0,p_1,x_1,\dots,x_{n-1}),\end{aligned}$$

satisfying, for every $(p,x)\in S^1\times S^{n-1}_+$,

$$\begin{aligned}(\psi^*\langle\!\langle\cdot,\cdot\rangle\!\rangle)_{(p,x)}&=\big((\mathrm{id}\times\pi)^*(\Phi^*\langle\!\langle\cdot,\cdot\rangle\!\rangle)\big)_x\\ &=\big((\mathrm{id}\times\pi)^*(-\pi(x)_1^2\,\mathrm{can}_{S^1}+\mathrm{can}_{H^{n-1}})\big)_x\\ &=-\pi(x)_1^2\,\mathrm{can}_{S^1}+\frac{1}{x_n^2}\,\mathrm{can}_{S^{n-1}_+}\\ &=\frac{1}{x_n^2}\big(-\mathrm{can}_{S^1}+\mathrm{can}_{S^{n-1}_+}\big).\end{aligned}\tag{3.14}$$

This concludes the proof. □

Following [O'Neill1983, Chap. 8, p. 228f], one defines the n-dimensional *anti-deSitter spacetime* $\widetilde{H}^n_1$ to be the universal covering manifold of the pseudohyperbolic space H^n_1. For $\widetilde{H}^n_1$ the sectional curvature is identically -1 and the scalar curvature equals $-n(n-1)$. In physics, $\widetilde{H}^4_1$ is important because it provides a vacuum solution to Einstein's field equation with cosmological constant $\Lambda = -3$.

The causality properties of $\widetilde{H}^n_1$ are discussed in [O'Neill1983, Chap. 14, Example 41]. It turns out that $\widetilde{H}^n_1$ is not globally hyperbolic. The conformal diffeomorphism constructed in Lemma 3.5.8 lifts to a conformal diffeomorphism of the universal covering manifolds:

$$\widetilde{\Psi}\colon \big(\mathbb{R}\times S^{n-1}_+, -dt^2 + \mathrm{can}_{S^{n-1}_+}\big) \to \big(\widetilde{H}^n_1, \langle\!\langle\cdot,\cdot\rangle\!\rangle\big)$$

such that for any $(t,x) \in \mathbb{R}^1 \times S^{n-1}_+ \subset \mathbb{R}^1 \times \mathbb{R}^n$ one has

$$\big(\psi^*\langle\!\langle\cdot,\cdot\rangle\!\rangle\big)_{(t,x)} = \frac{1}{x_n^2}\big(-dt^2 + \mathrm{can}_{S^{n-1}_+}\big).$$

Then $\widetilde{H}^n_1$ is conformally diffeomorphic to the causally compatible subset $\mathbb{R}\times S^{n-1}_+$ of the globally hyperbolic Einstein cylinder. From the considerations above we will derive existence of Green's operators for the Yamabe operator Y_g on anti-deSitter spacetime $\widetilde{H}^n_1$.

Definition 3.5.9. Let (M,g) be a Lorentzian manifold of dimension $n \geq 3$. Then the *Yamabe operator* Y_g acting on functions on M is given by

$$Y_g = 4\frac{n-1}{n-2}\,\Box_g + \mathrm{scal}_g \tag{3.15}$$

where $\Box_g$ denotes the d'Alembert operator and scal_g is the scalar curvature taken with respect to g.

We perform a conformal change of the metric. To simplify formulas we write the conformally related metric as $\tilde{g} = \varphi^{p-2}g$ where $p = \frac{2n}{n-2}$ and φ is a positive smooth function on M. The Yamabe operators for the metrics g and $\tilde{g}$ are related by

$$Y_{\tilde{g}}u = \varphi^{1-p}\cdot Y_g(\varphi u), \tag{3.16}$$

where $u \in C^\infty(M)$, see [Lee–Parker1987, p. 43, Eq. (2.7)]. Multiplying Y_g with $\frac{n-2}{4\cdot(n-1)}$ we obtain a normally hyperbolic operator

$$P_g = \Box_g + \frac{n-2}{4\cdot(n-1)}\cdot \mathrm{scal}_g.$$

Equation (3.16) gives for this operator

$$P_{\tilde{g}}u = \varphi^{1-p}\big(P_g(\varphi u)\big). \tag{3.17}$$

Now we consider this operator P_g on the Einstein cylinder $\mathbb{R} \times S^{n-1}$. Since the Einstein cylinder is globally hyperbolic we get unique advanced and retarded Green's operators $G_\pm$ for P_g. From Example 3.5.4 we know that $\mathbb{R} \times S_+^{n-1}$ is a causally compatible subset of the Einstein cylinder $\mathbb{R} \times S^{n-1}$. By Proposition 3.5.1 we have advanced and retarded Green's operators for P_g on $\mathbb{R} \times S_+^{n-1}$. From Equation (3.17) and Remark 3.5.6 we conclude

Corollary 3.5.10. *On the anti-deSitter spacetime $\widetilde{H}_1^n$ the Yamabe operator possesses advanced and retarded Green's operators.* □

Remark 3.5.11. It should be noted that the precise form of the zero order term of the Yamabe operator given by the scalar curvature is crucial for our argument. On $(\widetilde{H}_1^n, \tilde{g})$ the scalar curvature is constant, $\mathrm{scal}_{\tilde{g}} = -n(n-1)$. Hence the rescaled Yamabe operator is $P_{\tilde{g}} = \Box_{\tilde{g}} - \frac{1}{4}n(n-2) = \Box_{\tilde{g}} - c$ with $c := \frac{1}{4}n \cdot (n-2)$. For the d'Alembert operator $\Box_{\tilde{g}}$ on $(\widetilde{H}_1^n, \tilde{g})$ we have for any $u \in C^\infty(\widetilde{H}_1^n)$

$$\Box_{\tilde{g}} u = P_{\tilde{g}} u + c \cdot u = \varphi^{1-p} P_g(\varphi u) + c \cdot u = \varphi^{1-p}\big(P_g + c \cdot \varphi^{p-2}\big)(\varphi u).$$

The conformal factor φ^{p-2} tends to infinity as one approaches the boundary of $\mathbb{R} \times S_+^{n-1}$ in $\mathbb{R} \times S^{n-1}$. Namely, for $(t, x) \in \mathbb{R} \times S_+^{n-1}$ one has by (3.14) $\varphi^{p-2}(t, x) = x_n^{-2}$ where x_n denotes the last component of $x \in S^{n-1} \subset \mathbb{R}^n$. Hence if one approaches the boundary, then $x_n \to 0$ and therefore $\varphi^{p-2} = x_n^{-2} \to \infty$. Therefore one cannot extend the operator $P_g + c \cdot \varphi^{p-2}$ to an operator defined on the whole Einstein cylinder $\mathbb{R} \times S^{n-1}$. Thus we cannot establish existence of Green's operators for the d'Alembert operator on anti-deSitter spacetime with the methods developed here.

How about uniqueness of fundamental solutions for normally hyperbolic operators on anti-deSitter spacetime? We note that Theorem 3.1.1 cannot be applied for anti-deSitter spacetime because the time separation function τ is not finite. This can be seen as follows: We fix two points $x, y \in \mathbb{R} \times S_+^{n-1}$ with $x < y$ sufficiently far apart such that there exists a timelike curve connecting them in $\{(p, x) \in \mathbb{R} \times S^{n-1} \mid x_n \geq 0\}$ having a nonempty segment on the boundary $\{(p, x) \in \mathbb{R} \times S^{n-1} \mid x_n = 0\}$.

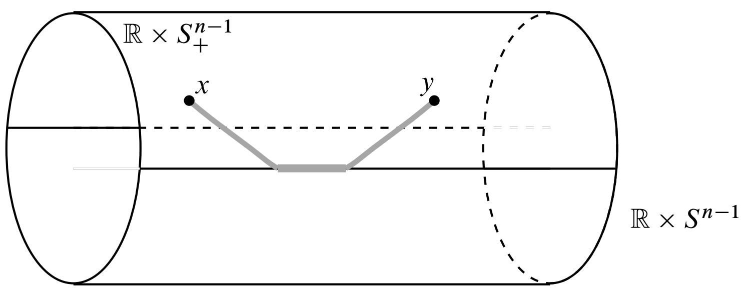


Figure 29. The time separation function is not finite on anti-deSitter spacetime.

By sliding the segment on the boundary slightly we obtain a timelike curve in the upper half of the Einstein cylinder connecting x and y whose length with respect to the metric $\frac{1}{x_n^2}(-\mathrm{can}_{S^1} + \mathrm{can}_{S_+^{n-1}})$ in (3.14) can be made arbitrarily large. This is due to the factor $\frac{1}{x_n^2}$ which is large if the segment is chosen so that x_n is small along it.

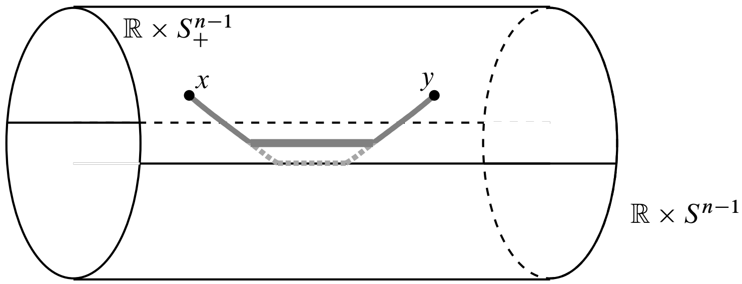


Figure 30. The time separation function is not finite on anti-deSitter spacetime.

A discussion as in Remark 3.1.5 considering supports (see picture below) shows that fundamental solutions for normally hyperbolic operators are not unique on the upper half $\mathbb{R} \times S_+^{n-1}$ of the Einstein cylinder. The fundamental solution of a point y in the lower half of the Einstein cylinder can be added to a given fundamental solution of x in the upper half thus yielding a second fundamental solution of x with the same support in the upper half.

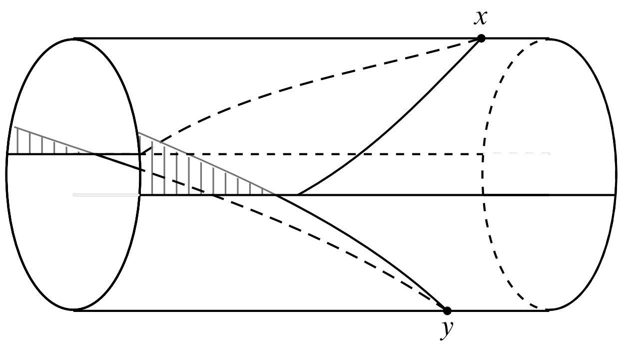


Figure 31. Advanced fundamental solution in x on the (open) upper half-cylinder is not unique.

Since anti-deSitter spacetime and $\mathbb{R} \times S_+^{n-1}$ are conformally equivalent we obtain distinct fundamental solutions for operators on anti-deSitter spacetime as described in Corollary 3.5.7.

4 Quantization

We now want to apply the analytical theory of wave equations and develop some mathematical basics of field (or second) quantization. We do not touch the so-called first quantization which is concerned with replacing point particles by wave functions. As in the preceding chapters we look at fields (sections in vector bundles) which have to satisfy some wave equation (specified by a normally hyperbolic operator) and now we want to quantize such fields.

We will explain two approaches. In the more traditional approach one constructs a quantum field which is a distribution satisfying the wave equation in the distributional sense. This quantum field takes its values in selfadjoint operators on Fock space which is the multi-particle space constructed out of the single-particle space of wave functions. This construction will however crucially depend on the choice of a Cauchy hypersurface.

It seems that for quantum field theory on curved spacetimes the approach of local quantum physics is more appropriate. The idea is to associate to each (reasonable) spacetime region the algebra of observables that can be measured in this region. We will find confirmed the saying that "quantization is a mystery, but second quantization is a functor" by mathematical physicist Edward Nelson. One indeed constructs a functor from the category of globally hyperbolic Lorentzian manifolds equipped with a formally selfadjoint normally hyperbolic operator to the category of C^*-algebras. We will see that this functor obeys the Haag–Kastler axioms of a local quantum field theory. This functorial interpretation of local covariant quantum field theory on curved spacetimes was introduced in [Hollands–Wald2001], [Verch2001], and [Brunetti–Fredenhagen–Verch2003].

It should be noted that in contrast to what is usually done in the physics literature there is no need to fix a wave equation and then quantize the corresponding fields (e.g. the Klein–Gordon field). In the present book, both the underlying manifold as well as the normally hyperbolic operator occur as variables in one single functor.

In Sections 4.1 and 4.2 we develop the theory of C^*-algebras and CCR-representations in full detail to the extent that we need. In the next three sections we construct the quantization functors and check the Haag–Kastler axioms. The last two sections are devoted to the construction of the Fock space and the quantum field. We will see that the quantum field determines the CCR-algebras up to isomorphism. This relates the two approaches to quantum field theory on curved backgrounds.

4.1 C^*-algebras

In this section we will collect those basic concepts and facts related to C^*-algebras that we will need when we discuss the canonical commutator relations in the subsequent section. We give complete proofs. Readers familiar with C^*-algebras may skip this section. For more information on C^*-algebras see e.g. [Bratteli–Robinson2002-I].

Definition 4.1.1. Let A be an associative $\mathbb{C}$-algebra, let $\|\cdot\|$ be a norm on the $\mathbb{C}$-vector space A, and let $*\colon A \to A$, $a \mapsto a^*$, be a $\mathbb{C}$-antilinear map. Then $(A, \|\cdot\|, *)$ is called a *C^*-algebra*, if $(A, \|\cdot\|)$ is complete and we have for all $a, b \in A$:

(1) $a^{**} = a$ ($*$ is an involution)

(2) $(ab)^* = b^* a^*$

(3) $\|ab\| \le \|a\|\,\|b\|$ (submultiplicativity)

(4) $\|a^*\| = \|a\|$ ($*$ is an isometry)

(5) $\|a^* a\| = \|a\|^2$ (C^*-property).

A (not necessarily complete) norm on A satisfying conditions (1) to (5) is called a *C^*-norm*.

Example 4.1.2. Let $(H, (\cdot\,,\cdot))$ be a complex Hilbert space, let $A = \mathcal{L}(H)$ be the algebra of bounded operators on H. Let $\|\cdot\|$ be the *operator norm*, i.e.,

$$\|a\| := \sup_{\substack{x \in H \\ \|x\|=1}} \|ax\|.$$

Let a^* be the operator adjoint to a, i.e.,

$$(ax, y) = (x, a^* y) \quad \text{for all } x,\ y \in H.$$

Axioms 1 to 4 are easily checked. Using Axioms 3 and 4 and the Cauchy–Schwarz inequality we see

$$\begin{aligned} \|a\|^2 &= \sup_{\|x\|=1} \|ax\|^2 = \sup_{\|x\|=1} (ax, ax) = \sup_{\|x\|=1} (x, a^* a x) \\ &\le \sup_{\|x\|=1} \|x\| \cdot \|a^* a x\| = \|a^* a\| \overset{\text{Axiom 3}}{\le} \|a^*\| \cdot \|a\| \overset{\text{Axiom 4}}{=} \|a\|^2. \end{aligned}$$

This shows Axiom 5.

Example 4.1.3. Let X be a locally compact Hausdorff space. Put

$$\begin{aligned} A &:= C_0(X) \\ &:= \{f : X \to \mathbb{C} \text{ continuous} \mid \text{ for all } \varepsilon > 0 \text{ there exists a compact subset} \\ &\qquad K \text{ of } X, \text{ so that } |f(x)| < \varepsilon \text{ for all } x \in X \setminus K\}. \end{aligned}$$

We call $C_0(X)$ the *algebra of continuous functions vanishing at infinity*. If X is compact, then $A = C_0(X) = C(X)$. All $f \in C_0(X)$ are bounded and we may define:

$$\|f\| := \sup_{x \in X} |f(x)|.$$

Moreover let

$$f^*(x) := \overline{f(x)}.$$

Then $(C_0(X), \|\cdot\|, *)$ is a commutative C^*-algebra.

Example 4.1.4. Let X be a differentiable manifold. Put

$$A := C_0^\infty(X) := C^\infty(X) \cap C_0(X).$$

We call $C_0^\infty(X)$ the *algebra of smooth functions vanishing at infinity*. Norm and $*$ are defined as in the previous example. Then $(C_0^\infty(X), \|\cdot\|, *)$ satisfies all axioms of a commutative C^*-algebra except that $(A, \|\cdot\|))$ is not complete. If we complete this normed vector space, then we are back to the previous example.

Definition 4.1.5. A subalgebra A_0 of a C^*-algebra A is called a *C^*-subalgebra* if it is a closed subspace and $a^* \in A_0$ for all $a \in A_0$.

Any C^*-subalgebra is a C^*-algebra in its own right.

Definition 4.1.6. Let S be a subset of a C^*-algebra A. Then the intersection of all C^*-subalgebras of A containing S is called the *C^*-subalgebra generated by S*.

Definition 4.1.7. An element a of a C^*-algebra is called *selfadjoint* if $a = a^*$.

Remark 4.1.8. Like any algebra a C^*-algebra A has at most one unit 1. Namely, let $1'$ be another unit, then

$$1 = 1 \cdot 1' = 1'.$$

Now we have for all $a \in A$

$$1^*a = (1^*a)^{**} = (a^*1^{**})^* = (a^*1)^* = a^{**} = a$$

and similarly one sees $a1^* = a$. Thus 1^* is also a unit. By uniqueness $1 = 1^*$, i.e., the unit is selfadjoint. Moreover,

$$\|1\| = \|1^*1\| = \|1\|^2,$$

hence $\|1\| = 1$ or $\|1\| = 0$. In the second case $1 = 0$ and therefore $A = 0$. Hence we may (and will) from now on assume that $\|1\| = 1$.

Example 4.1.9. (1) In Example 4.1.2 the algebra $A = \mathcal{L}(H)$ has a unit $1 = \mathrm{id}_H$.

(2)The algebra $A = C_0(X)$ has a unit $f \equiv 1$ if and only if $C_0(X) = C(X)$, i.e., if and only if X is compact.

Let A be a C^*-algebra with unit 1. We write $A^\times$ for the set of invertible elements in A. If $a \in A^\times$, then also $a^* \in A^\times$ because

$$a^* \cdot (a^{-1})^* = (a^{-1}a)^* = 1^* = 1,$$

and similarly $(a^{-1})^* \cdot a^* = 1$. Hence $(a^*)^{-1} = (a^{-1})^*$.

Lemma 4.1.10. *Let A be a C^*-algebra. Then the maps*

$$\begin{aligned} A\times A &\to A, & (a,b) &\mapsto a+b,\\ \mathbb{C}\times A &\to A, & (\alpha,a) &\mapsto \alpha a,\\ A\times A &\to A, & (a,b) &\mapsto a\cdot b,\\ A^\times &\to A^\times, & a &\mapsto a^{-1},\\ A &\to A, & a &\mapsto a^*, \end{aligned}$$

are continuous.

Proof. (a) The first two maps are continuous for all normed vector spaces. This easily follows from the triangle inequality and from homogeneity of the norm.

(b) *Continuity of multiplication.* Let $a_0, b_0 \in A$. Then we have for all $a, b \in A$ with $\|a-a_0\| < \varepsilon$ and $\|b-b_0\| < \varepsilon$:

$$\begin{aligned} \|ab-a_0b_0\| &= \|ab-a_0b+a_0b-a_0b_0\|\\ &\le \|a-a_0\|\cdot\|b\| + \|a_0\|\cdot\|b-b_0\|\\ &\le \varepsilon\big(\|b-b_0\|+\|b_0\|\big) + \|a_0\|\cdot\varepsilon\\ &\le \varepsilon\big(\varepsilon+\|b_0\|\big) + \|a_0\|\cdot\varepsilon. \end{aligned}$$

(c) *Continuity of inversion.* Let $a_0 \in A^\times$. Then we have for all $a \in A^\times$ with $\|a-a_0\| < \varepsilon < \|a_0^{-1}\|^{-1}$:

$$\begin{aligned} \|a^{-1}-a_0^{-1}\| &= \|a^{-1}(a_0-a)a_0^{-1}\|\\ &\le \|a^{-1}\|\cdot\|a_0-a\|\cdot\|a_0^{-1}\|\\ &\le \big(\|a^{-1}-a_0^{-1}\|+\|a_0^{-1}\|\big)\cdot\varepsilon\cdot\|a_0^{-1}\|. \end{aligned}$$

Thus

$$\underbrace{\big(1-\varepsilon\|a_0^{-1}\|\big)}_{>0,\text{ since }\varepsilon<\|a_0^{-1}\|^{-1}}\ \|a^{-1}-a_0^{-1}\| \le \varepsilon\cdot\|a_0^{-1}\|^2$$

and therefore

$$\|a^{-1}-a_0^{-1}\| \le \frac{\varepsilon}{1-\varepsilon\|a_0^{-1}\|}\cdot\|a_0^{-1}\|^2.$$

(d) *Continuity of* $*$ is clear because $*$ is an isometry. ☐

Remark 4.1.11. If $(A, \|\cdot\|, *)$ satisfies the axioms of a C^*-algebra except that $(A, \|\cdot\|)$ is not complete, then the above lemma still holds because completeness has not been used in the proof. Let $\bar{A}$ be the completion of A with respect to the norm $\|\cdot\|$. By the above lemma $+$, $\cdot$, and $*$ extend continuously to $\bar{A}$ thus making $\bar{A}$ into a C^*-algebra.

Definition 4.1.12. Let A be a C^*-algebra with unit 1. For $a \in A$ we call

$$r_A(a) := \{\lambda\in\mathbb{C} \mid \lambda\cdot 1 - a \in A^\times\}$$

the *resolvent set of a* and

$$\sigma_A(a) := \mathbb{C} \setminus r_A(a)$$

the *spectrum of a*. For $\lambda \in r_A(a)$

$$(\lambda \cdot 1 - a)^{-1} \in A$$

is called the *resolvent of a at* λ. Moreover, the number

$$\rho_A(a) := \sup\{|\lambda| \mid \lambda \in \sigma_A(a)\}$$

is called the *spectral radius of a*.

Example 4.1.13. Let X be a compact Hausdorff space and let $A = C(X)$. Then

$$\begin{aligned} A^\times &= \{f \in C(X) \mid f(x) \neq 0 \text{ for all } x \in X\}, \\ \sigma_{C(X)}(f) &= f(X) \subset \mathbb{C}, \\ r_{C(X)}(f) &= \mathbb{C} \setminus f(X), \\ \rho_{C(X)}(f) &= \|f\|_\infty = \max_{x \in X} |f(x)|. \end{aligned}$$

Proposition 4.1.14. *Let A be a C^*-algebra with unit* 1 *and let $a \in A$. Then $\sigma_A(a) \subset \mathbb{C}$ is a nonempty compact subset and the resolvent*

$$r_A(a) \to A, \quad \lambda \mapsto (\lambda \cdot 1 - a)^{-1},$$

is continuous. Moreover,

$$\rho_A(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} \leq \|a\|.$$

Proof. (a) Let $\lambda_0 \in r_A(a)$. For $\lambda \in \mathbb{C}$ with

$$|\lambda - \lambda_0| < \|(\lambda_0 1 - a)^{-1}\|^{-1} \tag{4.1}$$

the Neumann series

$$\sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1}$$

converges absolutely because

$$\begin{aligned} \|(\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1}\| &\leq |\lambda_0 - \lambda|^m \cdot \|(\lambda_0 1 - a)^{-1}\|^{m+1} \\ &= \|(\lambda_0 1 - a)^{-1}\| \cdot \underbrace{\left(\frac{\|(\lambda_0 1 - a)^{-1}\|}{|\lambda_0 - \lambda|^{-1}} \right)^m}_{<1 \text{ by } (4.1)}. \end{aligned}$$

Since A is complete the Neumann series converges in A. It converges to the resolvent $(\lambda 1 - a)^{-1}$ because

$$
\begin{aligned}
&(\lambda 1 - a) \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1} \\
&\quad = [(\lambda - \lambda_0) 1 + (\lambda_0 1 - a)] \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1} \\
&\quad = - \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^{m+1} (\lambda_0 1 - a)^{-m-1} + \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m} \\
&\quad = 1.
\end{aligned}
$$

Thus we have shown $\lambda \in r_A(a)$ for all λ satisfying (4.1). Hence $r_A(a)$ is open and $\sigma_A(a)$ is closed.

(b) *Continuity of the resolvent.* We estimate the difference of the resolvent of a at λ_0 and at λ using the Neumann series. If λ satisfies (4.1), then

$$
\begin{aligned}
\left\| (\lambda 1 - a)^{-1} - (\lambda_0 1 - a)^{-1} \right\| &= \left\| \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1} - (\lambda_0 1 - a)^{-1} \right\| \\
&\leq \sum_{m=1}^{\infty} |\lambda_0 - \lambda|^m \, \|(\lambda_0 1 - a)^{-1}\|^{m+1} \\
&= \|(\lambda_0 1 - a)^{-1}\| \cdot \frac{|\lambda_0 - \lambda| \cdot \|(\lambda_0 1 - a)^{-1}\|}{1 - |\lambda_0 - \lambda| \cdot \|(\lambda_0 1 - a)^{-1}\|} \\
&= |\lambda_0 - \lambda| \cdot \frac{\|(\lambda_0 1 - a)^{-1}\|^2}{1 - |\lambda_0 - \lambda| \cdot \|(\lambda_0 1 - a)^{-1}\|} \\
&\to 0 \quad \text{for } \lambda \to \lambda_0.
\end{aligned}
$$

Hence the resolvent is continuous.

(c) We show $\rho_A(a) \leq \inf_n \|a^n\|^{\frac{1}{n}} \leq \liminf_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. Let $n \in \mathbb{N}$ be fixed and let $|\lambda|^n > \|a^n\|$. Each $m \in \mathbb{N}_0$ can be written uniquely in the form $m = pn + q$, p, $q \in \mathbb{N}_0$, $0 \leq q \leq n - 1$. The series

$$
\frac{1}{\lambda} \sum_{m=0}^{\infty} \left(\frac{a}{\lambda}\right)^m = \frac{1}{\lambda} \sum_{q=0}^{n-1} \left(\frac{a}{\lambda}\right)^q \sum_{p=0}^{\infty} \Big(\underbrace{\frac{a^n}{\lambda^n}}_{\|\cdot\|<1} \Big)^p
$$

converges absolutely. Its limit is $(\lambda 1 - a)^{-1}$ because

$$
(\lambda 1 - a) \cdot \left(\sum_{m=0}^{\infty} \lambda^{-m-1} a^m \right) = \sum_{m=0}^{\infty} \lambda^{-m} a^m - \sum_{m=0}^{\infty} \lambda^{-m-1} a^{m+1} = 1
$$

and similarly

$$\Big(\sum_{m=0}^{\infty} \lambda^{-m-1} a^m\Big) \cdot (\lambda 1 - a) = 1.$$

Hence for $|\lambda|^n > \|a^n\|$ the element $(\lambda 1 - a)$ is invertible and thus $\lambda \in r_A(a)$. Therefore

$$\rho_A(a) \leq \inf_{n\in\mathbb{N}} \|a^n\|^{\frac{1}{n}} \leq \liminf_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

(d) We show $\rho_A(a) \geq \limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. We abbreviate $\widetilde{\rho}(a) := \limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}}$.

Case 1: $\widetilde{\rho}(a) = 0$. If a were invertible, then

$$1 = \|1\| = \|a^n a^{-n}\| \leq \|a^n\| \cdot \|a^{-n}\|$$

would imply $1 \leq \widetilde{\rho}(a) \cdot \widetilde{\rho}(a^{-1}) = 0$, which yields a contradiction. Therefore $a \notin A^\times$. Thus $0 \in \sigma_A(a)$. In particular, the spectrum of a is nonempty. Hence the spectral radius $\rho_A(a)$ is bounded from below by 0 and thus

$$\widetilde{\rho}(a) = 0 \leq \rho_A(a).$$

Case 2: $\widetilde{\rho}(a) > 0$. If $a_n \in A$ are elements for which $R_n := (1 - a_n)^{-1}$ exist, then

$$a_n \to 0 \quad \Leftrightarrow \quad R_n \to 1.$$

This follows from the fact that the map $A^\times \to A^\times$, $a \mapsto a^{-1}$, is continuous by Lemma 4.1.10. Put

$$S := \{\lambda \in \mathbb{C} \mid |\lambda| \geq \widetilde{\rho}(a)\}.$$

We want to show that $S \not\subset r_A(a)$ since then there exists $\lambda \in \sigma_A(a)$ such that $|\lambda| \geq \widetilde{\rho}(a)$ and hence

$$\rho_A(a) \geq |\lambda| \geq \widetilde{\rho}(a).$$

Assume in the contrary that $S \subset r_A(a)$. Let $\omega \in \mathbb{C}$ be an n-th root of unity, i.e., $\omega^n = 1$. For $\lambda \in S$ we also have $\frac{\lambda}{\omega^k} \in S \subset r_A(a)$. Hence there exists

$$\Big(\frac{\lambda}{\omega^k} 1 - a\Big)^{-1} = \frac{\omega^k}{\lambda}\Big(1 - \frac{\omega^k a}{\lambda}\Big)^{-1}$$

and we may define

$$R_n(a, \lambda) := \frac{1}{n} \sum_{k=1}^{n} \Big(1 - \frac{\omega^k a}{\lambda}\Big)^{-1}.$$

We compute

$$\begin{aligned}\Big(1-\frac{a^n}{\lambda^n}\Big)R_n(a,\lambda) &= \frac{1}{n}\sum_{k=1}^n\sum_{l=1}^n\Big(\frac{\omega^{k(l-1)}a^{l-1}}{\lambda^{l-1}}-\frac{\omega^{kl}a^l}{\lambda^l}\Big)\Big(1-\frac{\omega^k a}{\lambda}\Big)^{-1}\\ &= \frac{1}{n}\sum_{k=1}^n\sum_{l=1}^n\frac{\omega^{k(l-1)}a^{l-1}}{\lambda^{l-1}}\\ &= \frac{1}{n}\sum_{l=1}^n\frac{a^{l-1}}{\lambda^{l-1}}\underbrace{\sum_{k=1}^n(\omega^{l-1})^k}_{=\begin{cases}0 & \text{if } l\geq 2\\ n & \text{if } l=1\end{cases}}\\ &= 1.\end{aligned}$$

Similarly one sees $R_n(a,\lambda)\big(1-\frac{a^n}{\lambda^n}\big)=1$. Hence

$$R_n(a,\lambda)=\Big(1-\frac{a^n}{\lambda^n}\Big)^{-1}$$

for any $\lambda\in S\subset r_A(a)$. Moreover for $\lambda\in S$ we have

$$\begin{aligned}&\Big\|\Big(1-\frac{a^n}{\widetilde{\rho}(a)^n}\Big)^{-1}-\Big(1-\frac{a^n}{\lambda^n}\Big)^{-1}\Big\|\\ &\leq\frac{1}{n}\sum_{k=1}^n\Big\|\Big(1-\frac{\omega^k a}{\widetilde{\rho}(a)}\Big)^{-1}-\Big(1-\frac{\omega^k a}{\lambda}\Big)^{-1}\Big\|\\ &=\frac{1}{n}\sum_{k=1}^n\Big\|\Big(1-\frac{\omega^k a}{\widetilde{\rho}(a)}\Big)^{-1}\Big(1-\frac{\omega^k a}{\lambda}-1+\frac{\omega^k a}{\widetilde{\rho}(a)}\Big)\Big(1-\frac{\omega^k a}{\lambda}\Big)^{-1}\Big\|\\ &=\frac{1}{n}\sum_{k=1}^n\Big\|\Big(\frac{\widetilde{\rho}(a)}{\omega^k}1-a\Big)^{-1}\Big(-\frac{\widetilde{\rho}(a)a}{\omega^k}+\frac{\lambda a}{\omega^k}\Big)\Big(\frac{\lambda}{\omega^k}1-a\Big)^{-1}\Big\|\\ &\leq|\widetilde{\rho}(a)-\lambda|\cdot\|a\|\cdot\sup_{z\in S}\|(z1-a)^{-1}\|^2.\end{aligned}$$

The supremum is finite since $z\mapsto(z1-a)^{-1}$ is continuous on $r_A(a)\supset S$ by part (b) of the proof and since for $|z|\geq 2\cdot\|a\|$ we have

$$\|(z1-a)^{-1}\|\leq\frac{1}{|z|}\sum_{n=0}^\infty\underbrace{\frac{\|a\|^n}{|z|^n}}_{\leq(\frac{1}{2})^n}\leq\frac{2}{|z|}\leq\frac{1}{\|a\|}.$$

Outside the annulus $\overline{B}_{2\|a\|}(0)-B_{\widetilde{\rho}(a)}(0)$ the expression $\|(z1-a)^{-1}\|$ is bounded by $\frac{1}{\|a\|}$ and on the compact annulus it is bounded by continuity.

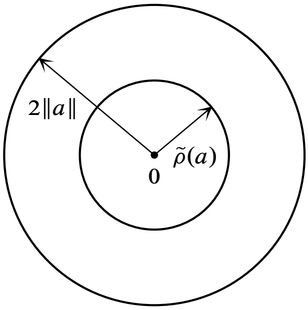


Figure 32. $\|(z1-a)^{-1}\|$ is bounded.

Put

$$C := \|a\| \cdot \sup_{z \in S} \|(z1-a)^{-1}\|^2.$$

We have shown

$$\|R_n(a,\widetilde{\rho}(a)) - R_n(a,\lambda)\| \le C \cdot |\widetilde{\rho}(a) - \lambda|$$

for all $n \in \mathbb{N}$ and all $\lambda \in S$. Putting $\lambda = \widetilde{\rho}(a) + \frac{1}{j}$ we obtain

$$\Big\|\Big(1 - \frac{a^n}{\widetilde{\rho}(a)^n}\Big)^{-1} - \underbrace{\Big(1 - \underbrace{\frac{a^n}{(\widetilde{\rho}(a)+\frac{1}{j})^n}}_{\to 0 \text{ for } n\to\infty}\Big)^{-1}}_{\to 1 \text{ for } n\to\infty}\Big\| \le \frac{C}{j},$$

thus

$$\limsup_{n\to\infty} \Big\|\Big(1 - \frac{a^n}{\widetilde{\rho}(a)^n}\Big)^{-1} - 1\Big\| \le \frac{C}{j}$$

for all $j \in \mathbb{N}$ and hence

$$\limsup_{n\to\infty} \Big\|\Big(1 - \frac{a^n}{\widetilde{\rho}(a)^n}\Big)^{-1} - 1\Big\| = 0.$$

For $n \to \infty$ we get

$$\Big(1 - \frac{a^n}{\widetilde{\rho}(a)^n}\Big)^{-1} \to 1$$

and thus

$$\frac{\|a^n\|}{\widetilde{\rho}(a)^n} \to 0. \tag{4.2}$$

On the other hand we have

$$\begin{aligned}
\|a^{n+1}\|^{\frac{1}{n+1}} &\le \|a\|^{\frac{1}{n+1}} \cdot \|a^n\|^{\frac{1}{n+1}} \\
&= \|a\|^{\frac{1}{n+1}} \cdot \|a^n\|^{-\frac{1}{n(n+1)}} \cdot \|a^n\|^{\frac{1}{n}} \\
&\le \|a\|^{\frac{1}{n+1}} \cdot \|a\|^{-\frac{n}{n(n+1)}} \cdot \|a^n\|^{\frac{1}{n}} \\
&= \|a^n\|^{\frac{1}{n}}.
\end{aligned}$$

Hence the sequence $\left(\|a^n\|^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ is monotonically nonincreasing and therefore

$$\widetilde{\rho}(a) = \limsup_{k\to\infty} \|a^k\|^{\frac{1}{k}} \leq \|a^n\|^{\frac{1}{n}} \quad \text{for all } n \in \mathbb{N}.$$

Thus $1 \leq \frac{\|a^n\|}{\widetilde{\rho}(a)^n}$ for all $n \in \mathbb{N}$, in contradiction to (4.2).

(e) *The spectrum is nonempty.* If $\sigma(a) = \emptyset$, then $\rho_A(a) = -\infty$ contradicting $\rho_A(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \geq 0$. □

Definition 4.1.15. Let A be a C^*-algebra with unit. Then $a \in A$ is called

- *normal*, if $aa^* = a^*a$,
- *an isometry*, if $a^*a = 1$, and
- *unitary*, if $a^*a = aa^* = 1$.

Remark 4.1.16. In particular, selfadjoint elements are normal. In a commutative algebra all elements are normal.

Proposition 4.1.17. *Let A be a C^*-algebra with unit and let $a \in A$. Then the following holds:*

(1) $\sigma_A(a^*) = \overline{\sigma_A(a)} = \{\lambda \in \mathbb{C} \mid \overline{\lambda} \in \sigma_A(a)\}$.
(2) *If $a \in A^\times$, then $\sigma_A(a^{-1}) = \sigma_A(a)^{-1}$.*
(3) *If a is normal, then $\rho_A(a) = \|a\|$.*
(4) *If a is an isometry, then $\rho_A(a) = 1$.*
(5) *If a is unitary, then $\sigma_A(a) \subset S^1 \subset \mathbb{C}$.*
(6) *If a is selfadjoint, then $\sigma_A(a) \subset [-\|a\|, \|a\|]$ and moreover $\sigma_A(a^2) \subset [0, \|a\|^2]$.*
(7) *If $P(z)$ is a polynomial with complex coefficients and $a \in A$ is arbitrary, then*

$$\sigma_A\big(P(a)\big) = P\big(\sigma_A(a)\big) = \{P(\lambda) \mid \lambda \in \sigma_A(a)\}.$$

Proof. We start by showing assertion (1). A number λ does not lie in the spectrum of a if and only if $(\lambda 1 - a)$ is invertible, i.e., if and only if $(\lambda 1 - a)^* = \overline{\lambda} 1 - a^*$ is invertible, i.e., if and only if $\overline{\lambda}$ does not lie in the spectrum of a^*.

To see (2) let a be invertible. Then 0 lies neither in the spectrum $\sigma_A(a)$ of a nor in the spectrum $\sigma_A(a^{-1})$ of a^{-1}. Moreover, we have for $\lambda \neq 0$

$$\lambda 1 - a = \lambda a(a^{-1} - \lambda^{-1} 1)$$

and

$$\lambda^{-1} 1 - a^{-1} = \lambda^{-1} a^{-1}(a - \lambda 1).$$

Hence $\lambda 1 - a$ is invertible if and only if $\lambda^{-1} 1 - a^{-1}$ is invertible.

To show (3) let a be normal. Then a^*a is selfadjoint, in particular normal. Using the C^*-property we obtain inductively

$$\begin{aligned}\|a^{2^n}\|^2 &= \|(a^{2^n})^* a^{2^n}\| = \|(a^*)^{2^n} a^{2^n}\| = \|(a^*a)^{2^n}\| \\ &= \|(a^*a)^{2^{n-1}}(a^*a)^{2^{n-1}}\| = \|(a^*a)^{2^{n-1}}\|^2 \\ &= \cdots = \|a^*a\|^{2^n} = \|a\|^{2^{n+1}}.\end{aligned}$$

Thus

$$\rho_A(a) = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \lim_{n\to\infty} \|a\| = \|a\|.$$

To prove (4) let a be an isometry. Then

$$\|a^n\|^2 = \|(a^n)^* a^n\| = \|(a^*)^n a^n\| = \|1\| = 1.$$

Hence

$$\rho_A(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 1.$$

For assertion (5) let a be unitary. On the one hand we have by (4)

$$\sigma_A(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \le 1\}.$$

On the other hand we have

$$\sigma_A(a) \overset{(1)}{=} \overline{\sigma_A(a^*)} = \overline{\sigma_A(a^{-1})} \overset{(2)}{=} \overline{\sigma_A(a)}^{-1}.$$

Both combined yield $\sigma_A(a) \subset S^1$.

To show (6) let a be selfadjoint. We need to show $\sigma_A(a) \subset \mathbb{R}$. Let $\lambda \in \mathbb{R}$ with $\lambda^{-1} > \|a\|$. Then $|-i\lambda^{-1}| = \lambda^{-1} > \rho(a)$ and hence $1 + i\lambda a = i\lambda(-i\lambda^{-1} + a)$ is invertible. Put

$$U := (1 - i\lambda a)(1 + i\lambda a)^{-1}.$$

Then $U^* = ((1+i\lambda a)^{-1})^*(1-i\lambda a)^* = (1-i\lambda a^*)^{-1}\cdot(1+i\lambda a^*) = (1-i\lambda a)^{-1}\cdot(1+i\lambda a)$ and therefore

$$\begin{aligned}U^*U &= (1-i\lambda a)^{-1}\cdot(1+i\lambda a)(1-i\lambda a)(1+i\lambda a)^{-1} \\ &= (1-i\lambda a)^{-1}(1-i\lambda a)(1+i\lambda a)(1+i\lambda a)^{-1} \\ &= 1.\end{aligned}$$

Similarly $UU^* = 1$, i.e., U is unitary. By (5) $\sigma_A(U) \subset S^1$. A simple computation with complex numbers shows that

$$|(1-i\lambda\mu)(1+i\lambda\mu)^{-1}| = 1 \quad \Leftrightarrow \quad \mu \in \mathbb{R}.$$

Thus $(1-i\lambda\mu)(1+i\lambda\mu)^{-1}\cdot 1 - U$ is invertible if $\mu \in \mathbb{C}\setminus\mathbb{R}$. From

$$\begin{aligned}&(1-i\lambda\mu)(1+i\lambda\mu)^{-1}\cdot 1 - U \\ &\quad= (1+i\lambda\mu)^{-1}\big((1-i\lambda\mu)(1+i\lambda a)1 - (1+i\lambda\mu)(1-i\lambda a)\big)(1+i\lambda a)^{-1} \\ &\quad= 2i\lambda(1+i\lambda\mu)^{-1}(a-\mu 1)(1+i\lambda\mu)^{-1}\end{aligned}$$

we see that $a - \mu 1$ is invertible for all $\mu \in \mathbb{C} \setminus \mathbb{R}$. Thus $\mu \in r_A(a)$ for all $\mu \in \mathbb{C} \setminus \mathbb{R}$ and hence $\sigma_A(a) \subset \mathbb{R}$. The statement about $\sigma_A(a^2)$ now follows from part (7).

Finally, to prove (7) decompose the polynomial $P(z) - \lambda$ into linear factors

$$P(z) - \lambda = \alpha \cdot \prod_{j=1}^{n} (\alpha_j - z), \quad \alpha, \alpha_j \in \mathbb{C}.$$

We insert an algebra element $a \in A$:

$$P(a) - \lambda 1 = \alpha \cdot \prod_{j=1}^{n} (\alpha_j 1 - a).$$

Since the factors in this product commute the product is invertible if and only if all factors are invertible.[1] In our case this means

$$\begin{aligned} \lambda \in \sigma_A\big(P(a)\big) &\Leftrightarrow \text{at least one factor is noninvertible} \\ &\Leftrightarrow \alpha_j \in \sigma_A(a) \text{ for some } j \\ &\Leftrightarrow \lambda = P(\alpha_j) \in P\big(\sigma_A(a)\big). \end{aligned}$$

□

Corollary 4.1.18. *Let $(A, \|\cdot\|, *)$ be a C^*-algebra with unit. Then the norm $\|\cdot\|$ is uniquely determined by A and $*$.*

Proof. For $a \in A$ the element a^*a is selfadjoint and hence by Proposition 4.1.17 (3)

$$\|a\|^2 = \|a^*a\| = \rho_A(a^*a)$$

depends only on A and $*$. □

Definition 4.1.19. Let A and B be C^*-algebras. An algebra homomorphism

$$\pi : A \to B$$

is called *$*$-morphism* if for all $a \in A$ we have

$$\pi(a^*) = \pi(a)^*.$$

A map $\pi : A \to A$ is called *$*$-automorphism* if it is an invertible $*$-morphism.

Corollary 4.1.20. *Let A and B be C^*-algebras with unit. Each unit-preserving $*$-morphism $\pi : A \to B$ satisfies*

$$\|\pi(a)\| \leq \|a\|$$

for all $a \in A$. In particular, π is continuous.

[1] This is generally true in algebras with unit. Let $b = a_1 \dots a_n$ with commuting factors. Then b is invertible if all factors are invertible: $b^{-1} = a_n^{-1} \dots a_1^{-1}$. Conversely, if b is invertible, then $a_i^{-1} = b^{-1} \cdot \prod_{j \neq i} a_j$ where we have used that the factors commute.

Proof. For $a \in A^\times$
$$\pi(a)\pi(a^{-1}) = \pi(aa^{-1}) = \pi(1) = 1$$
holds and similarly $\pi(a^{-1})\pi(a) = 1$. Hence $\pi(a) \in B^\times$ with $\pi(a)^{-1} = \pi(a^{-1})$. Now if $\lambda \in r_A(a)$, then

$$\lambda 1 - \pi(a) = \pi(\lambda 1 - a) \in \pi(A^\times) \subset B^\times,$$

i.e., $\lambda \in r_B(\pi(a))$. Hence $r_A(a) \subset r_B(\pi(a))$ and $\sigma_B(\pi(a)) \subset \sigma_A(a)$. This implies the inequality
$$\rho_B(\pi(a)) \leq \rho_A(a).$$
Since π is a $*$-morphism and a^*a and $\pi(a)^*\pi(a)$ are selfadjoint we can estimate the norm as follows:

$$\begin{aligned}\|\pi(a)\|^2 &= \|\pi(a)^*\pi(a)\| = \rho_B\big(\pi(a)^*\pi(a)\big) = \rho_B\big(\pi(a^*a)\big)\\ &\leq \rho_A(a^*a) = \|a\|^2.\end{aligned}$$ □

Corollary 4.1.21. *Let A be a C^*-algebra with unit. Then each unit-preserving $*$-automorphism $\pi: A \to A$ satisfies for all $a \in A$:*

$$\|\pi(a)\| = \|a\|.$$

Proof.
$$\|\pi(a)\| \leq \|a\| = \|\pi^{-1}\big(\pi(a)\big)\| \leq \|\pi(a)\|.$$ □

We extend Corollary 4.1.21 to the case where π is injective but not necessarily onto. This is not a direct consequence of Corollary 4.1.21 because it is not a priori clear that the image of a $*$-morphism is closed and hence a C^*-algebra in its own right.

Proposition 4.1.22. *Let A and B be C^*-algebras with unit. Each injective unit-preserving $*$-morphism $\pi: A \to B$ satisfies*

$$\|\pi(a)\| = \|a\|$$

for all $a \in A$.

Proof. By Corollary 4.1.20 we only have to show $\|\pi(a)\| \geq \|a\|$. Once we know this inequality for selfadjoint elements it follows for all $a \in A$ because

$$\|\pi(a)\|^2 = \|\pi(a)^*\pi(a)\| = \|\pi(a^*a)\| \geq \|a^*a\| = \|a\|^2.$$

Assume there exists a selfadjoint element $a \in A$ such that $\|\pi(a)\| < \|a\|$. By Proposition 4.1.17 $\sigma_A(a) \subset [-\|a\|, \|a\|]$ and $\rho_A(a) = \|a\|$, hence $\|a\| \in \sigma_A(a)$ or $-\|a\| \in \sigma_A(a)$. Similarly, $\sigma_B(\pi(a)) \subset [-\|\pi(a)\|, \|\pi(a)\|]$.

Choose a continuous function $f: [-\|a\|, \|a\|] \to \mathbb{R}$ such that f vanishes on $[-\|\pi(a)\|, \|\pi(a)\|]$ and $f(-\|a\|) = f(\|a\|) = 1$. By the Stone–Weierstrass theorem we can find polynomials P_n such that $\|f - P_n\|_{C^0([-\|a\|,\|a\|])} \to 0$ as $n \to \infty$. In particular, $\|P_n\|_{C^0([-\|\pi(a)\|,\|\pi(a)\|])} = \|f - P_n\|_{C^0([-\|\pi(a)\|,\|\pi(a)\|])} \to 0$ as $n \to \infty$. We may and will assume that the polynomials P_n are real.

From $\sigma_B(P_n(\pi(a))) = P_n(\sigma_B(\pi(a))) \subset P_n([-\|\pi(a)\|, \|\pi(a)\|])$ we see

$$\|P_n(\pi(a))\| = \rho_B(P_n(\pi(a))) \le \max |P_n([-\|\pi(a)\|, \|\pi(a)\|])| \xrightarrow{n\to\infty} 0$$

and thus

$$\lim_{n\to\infty} P_n(\pi(a)) = 0.$$

The sequence $(P_n(a))_n$ is a Cauchy sequence because

$$\begin{aligned}
\|P_n(a) - P_m(a)\| &= \rho_A(P_n(a) - P_m(a)) \\
&\le \max |(P_n - P_m)([-\|a\|, \|a\|])| \\
&= \|P_n - P_m\|_{C^0([-\|a\|,\|a\|])} \\
&\le \|P_n - f\|_{C^0([-\|a\|,\|a\|])} + \|f - P_m\|_{C^0([-\|a\|,\|a\|])}.
\end{aligned}$$

Denote its limit by $f(a) \in A$. Since $\|a\| \in \sigma_A(a)$ or $-\|a\| \in \sigma_A(a)$ and since $f(\pm\|a\|) = 1$ we have

$$\|f(a)\| = \lim_{n\to\infty} \|P_n(a)\| = \lim_{n\to\infty} \rho_B(P_n(a)) \ge \lim_{n\to\infty} |P_n(\pm\|a\|)| = 1.$$

Hence $f(a) \neq 0$. But $\pi(f(a)) = \pi(\lim_{n\to\infty} P_n(a)) = \lim_{n\to\infty} \pi(P_n(a)) = \lim_{n\to\infty} P_n(\pi(a)) = 0$. This contradicts the injectivity of π. □

4.2 The canonical commutator relations

In this section we introduce Weyl systems and CCR-representations. They formalize the "canonical commutator relations" from quantum field theory in an "exponentiated form" as we shall see later. The main result of the present section is Theorem 4.2.9 which says that for each symplectic vector space there is an essentially unique CCR-representation. Our approach follows ideas in [Manuceau1968]. A different proof of this result may be found in [Bratteli–Robinson2002-II, Sec. 5.2.2.2].

Let (V, ω) be a *symplectic vector space*, i.e., V is a real vector space of finite or infinite dimension and $\omega\colon V \times V \to \mathbb{R}$ is an antisymmetric bilinear map such that $\omega(\varphi, \psi) = 0$ for all $\psi \in V$ implies $\varphi = 0$.

Definition 4.2.1. A *Weyl system* of (V, ω) consists of a C^*-algebra A with unit and a map $W : V \to A$ such that for all $\varphi, \psi \in V$ we have

(i) $W(0) = 1$,

(ii) $W(-\varphi) = W(\varphi)^*$,

(iii) $W(\varphi) \cdot W(\psi) = e^{-i\omega(\varphi,\psi)/2}\, W(\varphi + \psi)$.

Condition (iii) says that W is a representation of the additive group V in A up to the "twisting factor" $e^{-i\omega(\varphi,\psi)/2}$. Note that since V is not given a topology there is no requirement on W to be continuous. In fact, we will see that even in the case when V is finite-dimensional and so V carries a canonical topology W will in general not be continuous.

Example 4.2.2. We construct a Weyl system for an arbitrary symplectic vector space (V, ω). Let $H = L^2(V, \mathbb{C})$ be the Hilbert space of square-integrable complex-valued functions on V with respect to the counting measure, i.e., H consists of those functions $F : V \to \mathbb{C}$ that vanish everywhere except for countably many points and satisfy

$$\|F\|_{L^2}^2 := \sum_{\varphi \in V} |F(\varphi)|^2 < \infty.$$

The Hermitian product on H is given by

$$(F, G)_{L^2} = \sum_{\varphi \in V} \overline{F(\varphi)} \cdot G(\varphi).$$

Let $A := \mathcal{L}(H)$ be the C^*-algebra of bounded linear operators on H as in Example 4.1.2. We define the map $W : V \to A$ by

$$(W(\varphi)F)(\psi) := e^{i\omega(\varphi,\psi)/2} F(\varphi + \psi).$$

Obviously $W(\varphi)$ is a bounded linear operator on H for any $\varphi \in V$ and $W(0) = \mathrm{id}_H = 1$. We check (ii) by making the substitution $\chi = \varphi + \psi$:

$$\begin{aligned}
(W(\varphi)F, G)_{L^2} &= \sum_{\psi \in V} \overline{(W(\varphi)F)(\psi)}\, G(\psi) \\
&= \sum_{\psi \in V} \overline{e^{i\omega(\varphi,\psi)/2} F(\varphi + \psi)}\, G(\psi) \\
&= \sum_{\chi \in V} \overline{e^{i\omega(\varphi,\chi-\varphi)/2} F(\chi)}\, G(\chi - \varphi) \\
&= \sum_{\chi \in V} \overline{e^{i\omega(\varphi,\chi)/2}} \cdot \overline{F(\chi)} \cdot G(\chi - \varphi) \\
&= \sum_{\chi \in V} \overline{F(\chi)} \cdot e^{i\omega(-\varphi,\chi)/2} \cdot G(\chi - \varphi) \\
&= (F, W(-\varphi)G)_{L^2}.
\end{aligned}$$

Hence $W(\varphi)^* = W(-\varphi)$. To check (iii) we compute

$$\begin{aligned}
(W(\varphi)(W(\psi)F))(\chi) &= e^{i\omega(\varphi,\chi)/2} (W(\psi)F)(\varphi + \chi) \\
&= e^{i\omega(\varphi,\chi)/2} e^{i\omega(\psi,\varphi+\chi)/2} F(\varphi + \chi + \psi) \\
&= e^{i\omega(\psi,\varphi)/2} e^{i\omega(\varphi+\psi,\chi)/2} F(\varphi + \chi + \psi) \\
&= e^{-i\omega(\varphi,\psi)/2} (W(\varphi + \psi)F)(\chi).
\end{aligned}$$

Thus $W(\varphi)W(\psi) = e^{-i\omega(\varphi,\psi)/2}\, W(\varphi+\psi)$. Let $\mathrm{CCR}(V,\omega)$ be the C^*-subalgebra of $\mathcal{L}(H)$ generated by the elements $W(\varphi)$, $\varphi \in V$. Then $\mathrm{CCR}(V,\omega)$ together with the map W forms a Weyl-system for (V,ω).

Proposition 4.2.3. *Let (A, W) be a Weyl system of a symplectic vector space (V,ω). Then*

(1) *$W(\varphi)$ is unitary for each $\varphi \in V$,*

(2) *$\|W(\varphi) - W(\psi)\| = 2$ for all $\varphi, \psi \in V$, $\varphi \neq \psi$,*

(3) *The algebra A is not separable unless $V = \{0\}$,*

(4) *the family $\{W(\varphi)\}_{\varphi\in V}$ is linearly independent.*

Proof. From $W(\varphi)^*W(\varphi) = W(-\varphi)\,W(\varphi) = e^{i\omega(-\varphi,\varphi)}W(0) = 1$ and similarly $W(\varphi)\,W(\varphi)^* = 1$ we see that $W(\varphi)$ is unitary.

To show (2) let $\varphi, \psi \in V$ with $\varphi \neq \psi$. For arbitrary $\chi \in V$ we have

$$\begin{aligned} W(\chi)\,W(\varphi-\psi)\,W(\chi)^{-1} &= W(\chi)\,W(\varphi-\psi)\,W(\chi)^* \\ &= e^{-i\omega(\chi,\varphi-\psi)/2}\,W(\chi+\varphi-\psi)\,W(-\chi) \\ &= e^{-i\omega(\chi,\varphi-\psi)/2}\,e^{-i\omega(\chi+\varphi-\psi,-\chi)/2}\,W(\chi+\varphi-\psi-\chi) \\ &= e^{-i\omega(\chi,\varphi-\psi)}\,W(\varphi-\psi). \end{aligned}$$

Hence the spectrum satisfies

$$\sigma_A(W(\varphi-\psi)) = \sigma_A(W(\chi)\,W(\varphi-\psi)\,W(\chi)^{-1}) = e^{-i\omega(\chi,\varphi-\psi)}\,\sigma_A(W(\varphi-\psi)).$$

Since $\varphi-\psi \neq 0$ the real number $\omega(\chi,\varphi-\psi)$ runs through all of $\mathbb{R}$ as χ runs through V. Therefore the spectrum of $W(\varphi-\psi)$ is U(1)-invariant. By Proposition 4.1.17 (5) the spectrum is contained in S^1 and by Proposition 4.1.14 it is nonempty. Hence $\sigma_A(W(\varphi-\psi)) = S^1$ and therefore

$$\sigma_A(e^{i\omega(\psi,\varphi)/2}\,W(\varphi-\psi)) = S^1.$$

Thus $\sigma_A(e^{i\omega(\psi,\varphi)/2}\,W(\varphi-\psi) - 1)$ is the circle of radius 1 centered at -1. Now Proposition 4.1.17 (3) says

$$\|e^{i\omega(\psi,\varphi)/2}\,W(\varphi-\psi) - 1\| = \rho_A\big(e^{i\omega(\psi,\varphi)/2}\,W(\varphi-\psi) - 1\big) = 2.$$

From $W(\varphi) - W(\psi) = W(\psi)(W(\psi)^*\,W(\varphi) - 1) = W(\psi)(e^{i\omega(\psi,\varphi)/2}\,W(\varphi-\psi) - 1)$ we conclude

$$\begin{aligned} &\|W(\varphi) - W(\psi)\|^2 \\ &\quad= \|(W(\varphi) - W(\psi))^*(W(\varphi) - W(\psi))\| \\ &\quad= \|(e^{i\omega(\psi,\varphi)/2}\,W(\varphi-\psi) - 1)^*\,W(\psi)^*\,W(\psi)\,(e^{i\omega(\psi,\varphi)/2}\,W(\varphi-\psi) - 1)\| \\ &\quad= \|(e^{i\omega(\psi,\varphi)/2}\,W(\varphi-\psi) - 1)^*\,(e^{i\omega(\psi,\varphi)/2}\,W(\varphi-\psi) - 1)\| \\ &\quad= \|e^{i\omega(\psi,\varphi)/2}\,W(\varphi-\psi) - 1\|^2 \\ &\quad= 4. \end{aligned}$$

This shows (2). Assertion (3) now follows directly since the balls of radius 1 centered at $W(\varphi)$, $\varphi \in V$, form an uncountable collection of mutually disjoint open subsets.

We now show (4). Let $\varphi_j \in V$, $j = 1, \dots, n$, be pairwise different and let $\sum_{j=1}^n \alpha_j W(\varphi_j) = 0$. We show $\alpha_1 = \cdots = \alpha_n = 0$ by induction on n. The case $n = 1$ is trivial by (1). Without loss of generality assume $\alpha_n \neq 0$. Hence

$$W(\varphi_n) = \sum_{j=1}^{n-1} \frac{-\alpha_j}{\alpha_n} W(\varphi_j)$$

and therefore

$$\begin{aligned} 1 &= W(\varphi_n)^* W(\varphi_n) \\ &= \sum_{j=1}^{n-1} \frac{-\alpha_j}{\alpha_n} W(-\varphi_n) W(\varphi_j) \\ &= \sum_{j=1}^{n-1} \frac{-\alpha_j}{\alpha_n} e^{-i\omega(-\varphi_n,\varphi_j)/2} W(\varphi_j - \varphi_n) \\ &= \sum_{j=1}^{n-1} \beta_j W(\varphi_j - \varphi_n) \end{aligned}$$

where we have put $\beta_j := \frac{-\alpha_j}{\alpha_n} e^{i\omega(\varphi_n,\varphi_j)/2}$. For an arbitrary $\psi \in V$ we obtain

$$\begin{aligned} 1 &= W(\psi) \cdot 1 \cdot W(-\psi) \\ &= \sum_{j=1}^{n-1} \beta_j W(\psi) W(\varphi_j - \varphi_n) W(-\psi) \\ &= \sum_{j=1}^{n-1} \beta_j e^{-i\omega(\psi,\varphi_j-\varphi_n)} W(\varphi_j - \varphi_n). \end{aligned}$$

From

$$\sum_{j=1}^{n-1} \beta_j W(\varphi_j - \varphi_n) = \sum_{j=1}^{n-1} \beta_j e^{-i\omega(\psi,\varphi_j-\varphi_n)} W(\varphi_j - \varphi_n)$$

we conclude by the induction hypothesis

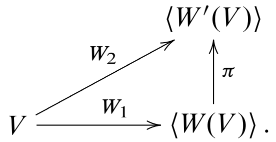
$$\beta_j = \beta_j e^{-i\omega(\psi,\varphi_j-\varphi_n)}$$

for all $j = 1, \dots, n-1$. If some $\beta_j \neq 0$, then $e^{-i\omega(\psi,\varphi_j-\varphi_n)} = 1$, hence

$$\omega(\psi, \varphi_j - \varphi_n) = 0$$

for all $\psi \in V$. Since ω is nondegenerate $\varphi_j - \varphi_n = 0$, a contradiction. Therefore all β_j and thus all α_j are zero, a contradiction. □

Remark 4.2.4. Let (A, W) be a Weyl system of the symplectic vector space (V, ω). Then the linear span of the $W(\varphi)$, $\varphi \in V$, is closed under multiplication and under $*$. This follows directly from the properties of a Weyl system. We denote this linear span by $\langle W(V)\rangle \subset A$. Now if (A', W') is another Weyl system of the same symplectic vector space (V, ω), then there is a unique linear map $\pi \colon \langle W(V)\rangle \to \langle W'(V)\rangle$ determined by $\pi(W(\varphi)) = W'(\varphi)$. Since π is given by a bijection on the bases $\{W(\varphi)\}_{\varphi\in V}$ and $\{W'(\varphi)\}_{\varphi\in V}$ it is a linear isomorphism. By the properties of a Weyl system π is a $*$-isomorphism. In other words, there is a unique $*$-isomorphism such that the following diagram commutes:



Remark 4.2.5. On $\langle W(V)\rangle$ we can define the norm

$$\Big\| \sum_{\varphi} a_\varphi W(\varphi) \Big\|_1 := \sum_{\varphi} |a_\varphi|.$$

This norm is not a C^*-norm but for every C^*-norm $\|\cdot\|_0$ on $\langle W(V)\rangle$ we have by the triangle inequality and by Proposition 4.2.3 (1)

$$\|a\|_0 \le \|a\|_1 \tag{4.3}$$

for all $a \in \langle W(V)\rangle$.

Lemma 4.2.6. *Let (A, W) be a Weyl system of a symplectic vector space (V, ω). Then*

$$\|a\|_{\max} := \sup\{\|a\|_0 \mid \|\cdot\|_0 \textit{ is a } C^*\textit{-norm on } \langle W(V)\rangle\}$$

defines a C^*-norm on $\langle W(V)\rangle$.

Proof. The given C^*-norm on A restricts to one on $\langle W(V)\rangle$, so the supremum is not taken on the empty set. Estimate (4.3) shows that the supremum is finite. The properties of a C^*-norm are easily checked. E. g. the triangle inequality follows from

$$\begin{aligned}
\|a+b\|_{\max} &= \sup\{\|a+b\|_0 \mid \|\cdot\|_0 \text{ is a } C^*\text{-norm on } \langle W(V)\rangle\} \\
&\le \sup\{\|a\|_0 + \|b\|_0 \mid \|\cdot\|_0 \text{ is a } C^*\text{-norm on } \langle W(V)\rangle\} \\
&\le \sup\{\|a\|_0 \mid \|\cdot\|_0 \text{ is a } C^*\text{-norm on } \langle W(V)\rangle\} \\
&\qquad + \sup\{\|b\|_0 \mid \|\cdot\|_0 \text{ is a } C^*\text{-norm on } \langle W(V)\rangle\} \\
&= \|a\|_{\max} + \|b\|_{\max}.
\end{aligned}$$

The other properties are shown similarly. □

Lemma 4.2.7. *Let (A, W) be a Weyl system of a symplectic vector space (V, ω). Then the completion $\overline{\langle W(V)\rangle}^{\max}$ of $\langle W(V)\rangle$ with respect to $\|\cdot\|_{\max}$ is simple, i.e., it has no nontrivial closed twosided $*$-ideals.*

Proof. By Remark 4.2.4 we may assume that (A, W) is the Weyl system constructed in Example 4.2.2. In particular, $\langle W(V)\rangle$ carries the C^*-norm $\|\cdot\|_{\mathrm{Op}}$, the operator norm given by $\langle W(V)\rangle \subset \mathcal{L}(H)$ where $H = L^2(V,\mathbb{C})$.

Let $I \subset \overline{\langle W(V)\rangle}^{\max}$ be a closed twosided $*$-ideal. Then $I_0 := I \cap \mathbb{C}\cdot W(0)$ is a (complex) vector subspace in $\mathbb{C}\cdot W(0) = \mathbb{C}\cdot 1 \cong \mathbb{C}$ and thus $I_0 = \{0\}$ or $I_0 = \mathbb{C}\cdot W(0)$. If $I_0 = \mathbb{C}\cdot W(0)$, then I contains 1 and therefore $I = \overline{\langle W(V)\rangle}^{\max}$. Hence we may assume $I_0 = \{0\}$.

Now we look at the projection map $P\colon \langle W(V)\rangle \to \mathbb{C}\cdot W(0)$, $P(\sum_\varphi a_\varphi W(\varphi)) = a_0 W(0)$. We check that P extends to a bounded operator on $\overline{\langle W(V)\rangle}^{\max}$. Let $\delta_0 \in L^2(V,\mathbb{C})$ denote the function given by $\delta_0(0) = 1$ and $\delta_0(\varphi) = 0$ otherwise. For $a = \sum_\varphi a_\varphi W(\varphi)$ and $\psi \in V$ we have

$$
\begin{aligned}
(a\cdot\delta_0)(\psi) &= \Big(\sum_\varphi a_\varphi W(\varphi)\delta_0\Big)(\psi) = \Big(\sum_\varphi a_\varphi\, e^{i\omega(\varphi,\psi)/2}\delta_0\Big)(\varphi+\psi)\\
&= a_{-\psi}\, e^{i\omega(-\psi,\psi)/2} = a_{-\psi}
\end{aligned}
$$

and therefore

$$
(\delta_0, a\cdot\delta_0)_{L^2} = \sum_{\psi\in V} \overline{\delta_0(\psi)}(a\cdot\delta_0)(\psi) = (a\cdot\delta_0)(0) = a_0.
$$

Moreover, $\|\delta_0\| = 1$. Thus

$$
\|P(a)\|_{\max} = \|a_0 W(0)\|_{\max} = |a_0| = |(\delta_0, a\cdot\delta_0)_{L^2}| \le \|a\|_{\mathrm{Op}} \le \|a\|_{\max}
$$

which shows that P extends to a bounded operator on $\overline{\langle W(V)\rangle}^{\max}$.

Now let $a \in I \subset \overline{\langle W(V)\rangle}^{\max}$. Fix $\varepsilon > 0$. We write

$$
a = a_0 W(0) + \sum_{j=1}^{n} a_j\, W(\varphi_j) + r
$$

where the $\varphi_j \neq 0$ are pairwise different and the remainder term r satisfies $\|r\|_{\max} < \varepsilon$. For any $\psi \in V$ we have

$$
I \ni W(\psi)\, a\, W(-\psi) = a_0 W(0) + \sum_{j=1}^{n} a_j\, e^{-i\omega(\psi,\varphi_j)}\, W(\varphi_j) + r(\psi)
$$

where $\|r(\psi)\|_{\max} = \|W(\psi) r W(-\psi)\|_{\max} \le \|r\|_{\max} < \varepsilon$. If we choose ψ_1 and ψ_2 such that $e^{-i\omega(\psi_1,\varphi_n)} = -e^{-i\omega(\psi_2,\varphi_n)}$, then adding the two elements

$$
\begin{aligned}
&a_0 W(0) + \sum_{j=1}^{n} a_j\, e^{-i\omega(\psi_1,\varphi_j)}\, W(\varphi_j) + r(\psi_1) \in I,\\
&a_0 W(0) + \sum_{j=1}^{n} a_j\, e^{-i\omega(\psi_2,\varphi_j)}\, W(\varphi_j) + r(\psi_2) \in I
\end{aligned}
$$

yields

$$a_0 W(0) + \sum_{j=1}^{n-1} a'_j\, W(\varphi_j) + r_1 \in I$$

where $\|r_1\|_{\max} = \left\|\frac{r(\psi_1)+r(\psi_2)}{2}\right\|_{\max} < \frac{\varepsilon+\varepsilon}{2} = \varepsilon$. Repeating this procedure we eventually get

$$a_0\, W(0) + r_n \in I$$

where $\|r_n\|_{\max} < \varepsilon$. Since ε is arbitrary and I is closed we conclude

$$P(a) = a_0\, W(0) \in I_0,$$

thus $a_0 = 0$.

For $a = \sum_\varphi a_\varphi\, W(\varphi) \in I$ and arbitrary $\psi \in V$ we have $W(\psi)a \in I$ as well, hence $P(W(\psi)a) = 0$. This means $a_{-\psi} = 0$ for all ψ, thus $a = 0$. This shows $I = \{0\}$. □

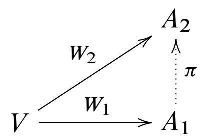
Definition 4.2.8. A Weyl system (A, W) of a symplectic vector space (V, ω) is called a *CCR-representation* of (V, ω) if A is generated as a C^*-algebra by the elements $W(\varphi)$, $\varphi \in V$. In this case we call A a *CCR-algebra* of (V, ω)

Of course, for any Weyl system (A, W) we can simply replace A by the C^*-subalgebra generated by the elements $W(\varphi)$, $\varphi \in V$, and we obtain a CCR-representation.

Existence of Weyl systems and hence CCR-representations has been established in Example 4.2.2. Uniqueness also holds in the appropriate sense.

Theorem 4.2.9. *Let (V, ω) be a symplectic vector space and let (A_1, W_1) and (A_2, W_2) be two CCR-representations of (V, ω).*

Then there exists a unique $*$-isomorphism $\pi : A_1 \to A_2$ such that the diagram



commutes.

Proof. We have to show that the $*$-isomorphism $\pi : \langle W_1(V)\rangle \to \langle W_2(V)\rangle$ as constructed in Remark 4.2.4 extends to an isometry $(A_1, \|\cdot\|_1) \to (A_2, \|\cdot\|_2)$. Since the pull-back of the norm $\|\cdot\|_2$ on A_2 to $\langle W_1(V)\rangle$ via π is a C^*-norm we have $\|\pi(a)\|_2 \le \|a\|_{\max}$ for all $a \in \langle W_1(V)\rangle$. Hence π extends to a $*$-morphism $\overline{\langle W_1(V)\rangle}^{\max} \to A_2$. By Lemma 4.2.7 the kernel of π is trivial, hence π is injective. Proposition 4.1.22 implies that $\pi : (\overline{\langle W_1(V)\rangle}^{\max}, \|\cdot\|_{\max}) \to (A_2, \|\cdot\|_2)$ is an isometry.

In the special case $(A_1, \|\cdot\|_1) = (A_2, \|\cdot\|_2)$ where π is the identity this yields $\|\cdot\|_{\max} = \|\cdot\|_1$. Thus for arbitrary A_2 the map π extends to an isometry $(A_1, \|\cdot\|_1) \to (A_2, \|\cdot\|_2)$. □

From now on we will call $\mathrm{CCR}(V,\omega)$ as defined in Example 4.2.2 *the* CCR-algebra of (V,ω).

Corollary 4.2.10. *CCR-algebras of symplectic vector spaces are simple, i.e., all unit preserving $*$-morphisms to other C^*-algebras are injective.*

Proof. Direct consequence of Corollary 4.1.20 and Lemma 4.2.7. □

Corollary 4.2.11. *Let (V_1,ω_1) and (V_2,ω_2) be two symplectic vector spaces and let $S\colon V_1 \to V_2$ be a symplectic linear map, i.e., $\omega_2(S\varphi, S\psi) = \omega_1(\varphi,\psi)$ for all $\varphi,\psi \in V_1$.*

Then there exists a unique injective $*$-morphism

$$\mathrm{CCR}(S)\colon \mathrm{CCR}(V_1,\omega_1) \to \mathrm{CCR}(V_2,\omega_2)$$

such that the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\quad S \quad} & V_2 \\ {\scriptstyle W_1}\downarrow & & \downarrow{\scriptstyle W_2} \\ \mathrm{CCR}(V_1,\omega_1) & \xrightarrow{\ \mathrm{CCR}(S)\ } & \mathrm{CCR}(V_2,\omega_2) \end{array}$$

commutes.

Proof. One immediately sees that $(\mathrm{CCR}(V_2,\omega_2), W_2 \circ S)$ is a Weyl system of (V_1,ω_1). Theorem 4.2.9 yields the result. □

From uniqueness of the map $\mathrm{CCR}(S)$ we conclude that $\mathrm{CCR}(\mathrm{id}_V) = \mathrm{id}_{\mathrm{CCR}(V,\omega)}$ and $\mathrm{CCR}(S_2 \circ S_1) = \mathrm{CCR}(S_2) \circ \mathrm{CCR}(S_1)$. In other words, we have constructed a functor

$$\mathrm{CCR}\colon \mathsf{SymplVec} \to \mathsf{C^*\text{-}Alg}$$

where $\mathsf{SymplVec}$ denotes the category whose objects are symplectic vector spaces and whose morphisms are symplectic linear maps, i.e., linear maps $A\colon (V_1,\omega_1) \to (V_2,\omega_2)$ with $A^*\omega_2 = \omega_1$. By $\mathsf{C^*\text{-}Alg}$ we denote the category whose objects are C^*-algebras and whose morphisms are *injective* unit preserving $*$-morphisms. Observe that symplectic linear maps are automatically injective.

In the case $V_1 = V_2$ the induced $*$-automorphisms $\mathrm{CCR}(S)$ are called *Bogoliubov transformation* in the physics literature.

4.3 Quantization functors

In the preceding section we introduced the functor CCR from the category $\mathsf{SymplVec}$ of symplectic vector spaces (with symplectic linear maps as morphisms) to the category $\mathsf{C^*\text{-}Alg}$ of C^*-algebras (with unit preserving $*$-monomorphisms as morphisms). We want to link these considerations to Lorentzian manifolds and the analysis of normally hyperbolic operators. In order to achieve this we introduce two further categories which are of geometric-analytical nature.

So far we have treated real and complex vector bundles E over the manifold M on an equal footing. From now on we will restrict ourselves to real bundles. This is not very restrictive since we can always forget a complex structure and regard complex bundles as real bundles. We will have to give the real bundle E another piece of additional structure. We will assume that E comes with a *nondegenerate inner product* $\langle\cdot,\cdot\rangle$. This means that each fiber E_x is equipped with a nondegenerate symmetric bilinear form depending smoothly on the base point x. In other words, $\langle\cdot,\cdot\rangle$ is like a Riemannian metric except that it need not be positive definite.

We say that a differential operator P acting on sections in E is *formally selfadjoint* with respect to the inner product $\langle\cdot,\cdot\rangle$ of E, if

$$\int_M \langle P\varphi, \psi\rangle \, \mathrm{dV} = \int_M \langle \varphi, P\psi\rangle \, \mathrm{dV}$$

for all $\varphi, \psi \in \mathcal{D}(M, E)$.

Example 4.3.1. Let M be an n-dimensional time-oriented connected Lorentzian manifold with metric g. Let E be a real vector bundle over M with inner product $\langle\cdot,\cdot\rangle$. Let ∇ be a connection on E. We assume that ∇ is *metric* with respect to $\langle\cdot,\cdot\rangle$, i.e.,

$$\partial_X\langle\varphi,\psi\rangle = \langle\nabla_X\varphi,\psi\rangle + \langle\varphi,\nabla_X\psi\rangle$$

for all sections $\varphi, \psi \in C^\infty(M, E)$. The inner product induces an isomorphism[2] $\text{☞}: E \to E^*$, $\varphi \mapsto \langle\varphi,\cdot\rangle$. Since ∇ is metric we get

$$\begin{aligned}(\nabla_X(\text{☞}\varphi))\cdot\psi &= \partial_X((\text{☞}\varphi)\psi) - (\text{☞}\varphi)(\nabla_X\psi) = \partial_X\langle\varphi,\psi\rangle - \langle\varphi,\nabla_X\psi\rangle \\ &= \langle\nabla_X\varphi,\psi\rangle = (\text{☞}(\nabla_X\varphi))\cdot\psi,\end{aligned}$$

hence

$$\nabla_X(\text{☞}\varphi) = \text{☞}(\nabla_X\varphi).$$

Equation (3.2) says for all $\varphi, \psi \in C^\infty(M, E)$

$$(\text{☞}\varphi)\cdot(\Box^\nabla\psi) = \sum_{i=1}^n \varepsilon_i \nabla_{e_i}(\text{☞}\varphi)\cdot\nabla_{e_i}\psi - \mathrm{div}(V_1),$$

thus

$$\langle\varphi, \Box^\nabla\psi\rangle = \sum_{i=1}^n \varepsilon_i \langle\nabla_{e_i}\varphi, \nabla_{e_i}\psi\rangle - \mathrm{div}(V_1),$$

where V_1 is a smooth vector field with $\mathrm{supp}(V_1) \subset \mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi)$ and $e_1, \ldots, e_n$ is a local Lorentz orthonormal tangent frame, $\varepsilon_i = g(e_i, e_i)$. Interchanging the roles

[2] In case $E = TM$ with the Lorentzian metric as the inner product we write $\flat$ instead of ☞ and $\sharp$ instead of ☞^{-1}. Hence if $E = T^*M$ we have $\text{☞} = \sharp$ and $\text{☞}^{-1} = \flat$.

of φ and ψ we get

$$\langle \Box^\nabla \varphi, \psi \rangle = \sum_{i=1}^n \varepsilon_i \langle \nabla_{e_i} \varphi, \nabla_{e_i} \psi \rangle - \mathrm{div}(V_2),$$

and therefore

$$\langle \varphi, \Box^\nabla \psi \rangle - \langle \Box^\nabla \varphi, \psi \rangle = \mathrm{div}(V_2 - V_1).$$

If $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi)$ is compact we obtain

$$\int_M \langle \varphi, \Box^\nabla \psi \rangle \, \mathrm{dV} - \int_M \langle \Box^\nabla \varphi, \psi \rangle \, \mathrm{dV} = \int_M \mathrm{div}(V_2 - V_1) \, \mathrm{dV} = 0,$$

thus $\Box^\nabla$ is formally selfadjoint. If, moreover, $B \in C^\infty(M, \mathrm{End}(E))$ is selfadjoint with respect to $\langle \cdot\, , \cdot \rangle$, then the normally hyperbolic operator $P = \Box^\nabla + B$ is formally selfadjoint.

As a special case let E be the trivial real line bundle. In other words, sections in E are simply real-valued functions. The inner product is given by the pointwise product. In this case the inner product is positive definite. Then the above discussion shows that the d'Alembert operator $\Box$ is formally selfadjoint and, more generally, $P = \Box + B$ is formally selfadjoint where the zero-order term B is a smooth real-valued function on M. This includes the (normalized) Yamabe operator P_g discussed in Section 3.5, the *Klein–Gordon operator* $P = \Box + m^2$ and the *covariant Klein–Gordon operator* $P = \Box + m^2 + \kappa\,\mathrm{scal}$, where m and κ are real constants.

Example 4.3.2. Let M be an n-dimensional time-oriented connected Lorentzian manifold. Let $\Lambda^k T^*M$ be the bundle of k-forms on M. The Lorentzian metric induces a nondegenerate inner product $\langle \cdot\, , \cdot \rangle$ on $\Lambda^k T^*M$, which is indefinite if $1 \le k \le n-1$. Let $d\colon C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ denote exterior differentiation. Let $\delta\colon C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k-1} T^*M)$ be the *codifferential*. This is the unique differential operator formally adjoint to d, i.e.,

$$\int_M \langle d\varphi, \psi \rangle \, \mathrm{dV} = \int_M \langle \varphi, \delta\psi \rangle \, \mathrm{dV}$$

for all $\varphi \in \mathcal{D}(M, \Lambda^k T^*M)$ and $\psi \in \mathcal{D}(M, \Lambda^{k+1} T^*M)$. Then the operator

$$P = d\delta + \delta d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^k T^*M)$$

is obviously formally selfadjoint. The Levi-Civita connection induces a metric connection ∇ on the bundle $\Lambda^k T^*M$. The *Weitzenböck formula* relates P and $\Box^\nabla$, $P = \Box^\nabla + B$ where B is a certain expression in the curvature tensor of M, see [Besse1987, Eq. (12.92′)]. In particular, P and $\Box^\nabla$ have the same principal symbol, hence P is normally hyperbolic.

The operator P appears in physics in different contexts. Let M be of dimension $n = 4$. Let us first look at the *Proca equation* describing a spin-1 particle of mass

$m > 0$. The quantum mechanical wave function of such a particle is given by $A \in C^\infty(M, \Lambda^1 T^*M)$ and satisfies

$$\delta dA + m^2 A = 0. \tag{4.4}$$

Applying δ to this equation and using $\delta^2 = 0$ and $m \neq 0$ we conclude $\delta A = 0$. Thus the Proca equation (4.4) is equivalent to

$$(P + m^2)A = 0$$

together with the constraint $\delta A = 0$.

Now we discuss *electrodynamics*. Let M be a 4-dimensional globally hyperbolic Lorentzian manifold and assume that the second deRham cohomology vanishes, $H^2(M; \mathbb{R}) = \{0\}$. By Poincaré duality, the second cohomology with compact supports also vanishes, $H^2_c(M; \mathbb{R}) = \{0\}$. See [Warner1983] for details on deRham cohomology.

The electric and the magnetic fields can be combined to the *field strength* $F \in \mathcal{D}(M, \Lambda^2 T^*M)$. The *Maxwell equations* are

$$dF = 0 \quad \text{and} \quad \delta F = J$$

where $J \in \mathcal{D}(M, \Lambda^1 T^*M)$ is the *current density*. From $H^2_c(M; \mathbb{R}) = 0$ we have that $dF = 0$ implies the existence of a *vector potential* $A \in \mathcal{D}(M, \Lambda^1 T^*M)$ with $dA = F$. Now $\delta A \in \mathcal{D}(M, \mathbb{R})$ and by Theorem 3.2.11 we can find $f \in C^\infty_{\mathrm{sc}}(M, \mathbb{R})$ with $\Box f = \delta A$. We put $A' := A - df$. We see that $dA' = dA = F$ and $\delta A' = \delta A - \delta df = \delta A - \Box f = 0$. A vector potential satisfying the last equation $\delta A' = 0$ is said to be in *Lorentz gauge*. From the Maxwell equations we conclude $\delta dA' = \delta F = J$. Hence

$$PA' = J.$$

Example 4.3.3. Next we look at spinors and the Dirac operator. These concepts are studied in much detail on general semi-Riemannian manifolds in [Baum1981], see also [Bär–Gauduchon–Moroianu2005, Sec. 2] for an overview.

Let M be an n-dimensional oriented and time-oriented connected Lorentzian manifold. Furthermore, we assume that M carries a spin structure. Then we can form the *spinor bundle* ΣM over M. This is a complex vector bundle of rank $2^{n/2}$ or $2^{(n-1)/2}$ depending on whether n is even or odd. This bundle carries an indefinite Hermitian product h.

The *Dirac operator* $D\colon C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M)$ is a formally selfadjoint differential operator of first order. The Levi-Civita connection induces a metric connection ∇ on ΣM. The Weitzenböck formula says

$$D^2 = \Box^\nabla + \frac{1}{4}\mathrm{scal}.$$

Thus D^2 is normally hyperbolic. Since D is formally selfadjoint so is D^2.

If we forget the complex structure on ΣM, i.e., we regard ΣM as a real bundle, and if we let $\langle \cdot , \cdot \rangle$ be given by the real part of h, then the operator $P = D^2$ is of the type under consideration in this section.

Now we define the category of globally hyperbolic manifolds equipped with normally hyperbolic operators:

Definition 4.3.4. The category GlobHyp is defined as follows: The objects of GlobHyp are triples (M, E, P) where M is a globally hyperbolic Lorentzian manifold, $E \to M$ is a real vector bundle with nondegenerate inner product, and P is a formally selfadjoint normally hyperbolic operator acting on sections in E.

A morphism $(M_1, E_1, P_1) \to (M_2, E_2, P_2)$ in GlobHyp is a pair (f, F) where $f : M_1 \to M_2$ is a time-orientation preserving isometric embedding so that $f(M_1) \subset M_2$ is a causally compatible open subset. Moreover, $F : E_1 \to E_2$ is a vector bundle homomorphism over f which is fiberwise an isometry. In particular,

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

commutes. Furthermore, F has to preserve the normally hyperbolic operators, i.e.,

$$\begin{array}{ccc} \mathcal{D}(M_1, E_1) & \xrightarrow{P_1} & \mathcal{D}(M_1, E_1) \\ {\scriptstyle \mathrm{ext}}\downarrow & & \downarrow{\scriptstyle \mathrm{ext}} \\ \mathcal{D}(M_2, E_2) & \xrightarrow{P_2} & \mathcal{D}(M_2, E_2) \end{array}$$

commutes where $\mathrm{ext}(\varphi)$ denotes the extension of $F \circ \varphi \circ f^{-1} \in \mathcal{D}(f(M_1), E_2)$ to all of M_2 by 0.

Notice that a morphism between two objects (M_1, E_1, P_1) and (M_2, E_2, P_2) can exist only if M_1 and M_2 have equal dimension and if E_1 and E_2 have the same rank. The condition that $f(M_1) \subset M_2$ is causally compatible does not follow from the fact that M_1 and M_2 are globally hyperbolic. For example, consider $M_2 = \mathbb{R} \times S^1$ with metric $-dt^2 + \mathrm{can}_{S^1}$ and let $M_1 \subset M_2$ be a small strip about a spacelike helix. Then M_1 and M_2 are both intrinsically globally hyperbolic but M_1 is not a causally compatible subset of M_2.

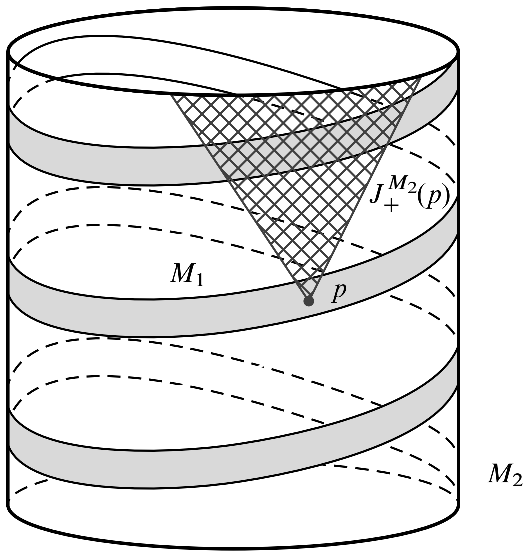


Figure 33. The spacelike helix M_1 is not causally compatible in M_2.

Lemma 4.3.5. *Let M be a globally hyperbolic Lorentzian manifold. Let $E \to M$ be a real vector bundle with nondegenerate inner product $\langle\cdot\,,\cdot\rangle$. Consider a formally selfadjoint normally hyperbolic operator P with advanced and retarded Green's operators $G_\pm$ as in Corollary 3.4.3. Then*

$$\int_M \langle G_\pm\varphi, \psi\rangle \,\mathrm{dV} = \int_M \langle \varphi, G_\mp\psi\rangle \,\mathrm{dV} \tag{4.5}$$

holds for all $\varphi, \psi \in \mathcal{D}(M, E)$.

Proof. The proof is basically the same as that of Lemma 3.4.4. Namely, for Green's operators we have $PG_\pm = \mathrm{id}_{\mathcal{D}(M,E)}$ and therefore we get

$$\begin{aligned}\int_M \langle G_\pm\varphi, \psi\rangle \,\mathrm{dV} &= \int_M \langle G_\pm\varphi, PG_\mp\psi\rangle \,\mathrm{dV} \\ &= \int_M \langle PG_\pm\varphi, G_\mp\psi\rangle \,\mathrm{dV} = \int_M \langle \varphi, G_\mp\psi\rangle \,\mathrm{dV}\end{aligned}$$

where we have made use of the formal selfadjointness of P in the second equality. Notice that $\mathrm{supp}(G_\pm\varphi) \cap \mathrm{supp}(G_\mp\psi) \subset J_\pm^M(\mathrm{supp}(\varphi)) \cap J_\mp^M(\mathrm{supp}(\psi))$ is compact in a globally hyperbolic manifold so that the partial integration is justified.

Alternatively, we can also argue as follows. The inner product yields the vector bundle isomorphism ☞: $E \to E^*$, $e \mapsto \langle e, \cdot\rangle$, as noted in Example 4.3.1. Formal selfadjointness now means that the operator P corresponds to the dual operator P^* under ☞. Now Lemma 4.3.5 is a direct consequence of Lemma 3.4.4. $\square$

If we want to deal with Lorentzian manifolds which are not globally hyperbolic we have the problem that Green's operators need not exist and if they do they are in general no longer unique. In this case we have to provide the Green's operators as additional data. This motivates the definition of a category of Lorentzian manifolds with normally hyperbolic operators and global fundamental solutions.

Definition 4.3.6. Let LorFund denote the category whose objects are 5-tuples (M, E, P, G_+, G_-) where M is a time-oriented connected Lorentzian manifold, E is a real vector bundle over M with nondegenerate inner product, P is a formally self-adjoint normally hyperbolic operator acting on sections in E, and $G_\pm$ are advanced and retarded Green's operators for P respectively. Moreover, we *assume* that (4.5) holds for all $\varphi, \psi \in \mathcal{D}(M, E)$.

Let $X = (M_1, E_1, P_1, G_{1,+}, G_{1,-})$ and $Y = (M_2, E_2, P_2, G_{2,+}, G_{2,-})$ be two objects in LorFund. If M_1 is not globally hyperbolic, then we let the set of morphisms from X to Y be empty unless $X = Y$ in which case we put $\mathrm{Mor}(X, Y) := \{(\mathrm{id}_{M_1}, \mathrm{id}_{E_1})\}$.

If M_1 is globally hyperbolic, then $\mathrm{Mor}(X, Y)$ consists of all pairs (f, F) with the same properties as those of the morphisms in GlobHyp. It then follows from Proposition 3.5.1 and Corollary 3.4.3 that we automatically have compatibility of the Green's operators, i.e., the diagram

$$
\begin{array}{ccc}
\mathcal{D}(M_1, E_1) & \xrightarrow{\ \mathrm{ext}\ } & \mathcal{D}(M_2, E_2) \\
\Big\downarrow {\scriptstyle G_{1,\pm}} & & \Big\downarrow {\scriptstyle G_{2,\pm}} \\
C^\infty(M_1, E_1) & \xleftarrow{\ \mathrm{res}\ } & C^\infty(M_2, E_2)
\end{array}
$$

commutes. Here res stands for "restriction". More precisely, $\mathrm{res}(\varphi) = F^{-1} \circ \varphi \circ f$. Composition of morphisms is given by the usual composition of maps.

The definition of the category LorFund is such that nontrivial morphisms exist only if the source manifold M_1 is globally hyperbolic while there is no such restriction on the target manifold M_2. It will become clear in the proof of Lemma 4.3.8 why we restrict to globally hyperbolic M_1.

By Corollary 3.4.3 there exist unique advanced and retarded Green's operators $G_\pm$ for a normally hyperbolic operator on a globally hyperbolic manifold. Hence we can define

$$\mathrm{SOLVE}(M, E, P) := (M, E, P, G_+, G_-)$$

on objects of GlobHyp and

$$\mathrm{SOLVE}(f, F) := (f, F)$$

on morphisms.

Lemma 4.3.7. *This defines a functor* SOLVE: GlobHyp → LorFund.

Proof. We only need to check that $\mathrm{SOLVE}(f, F) = (f, F)$ is actually a morphism in LorFund, i.e., that (f, F) is compatible with the Green's operators. By uniqueness of Green's operators on globally hyperbolic manifolds it suffices to show that $\mathrm{res} \circ G_{2,+} \circ \mathrm{ext}$ is an advanced Green's operator on M_1 and similarly for $G_{2,-}$. Since $f(M_1) \subset M_2$ is a causally compatible connected open subset this follows from Proposition 3.5.1. □

Next we would like to use the Green's operators in order to construct a symplectic vector space to which we can then apply the functor CCR. Let (M, E, P, G_+, G_-) be an object of LorFund. Using $G = G_+ - G_-: \mathcal{D}(M, E) \to C^\infty(M, E)$ we define

$$\tilde{\omega}: \mathcal{D}(M, E) \times \mathcal{D}(M, E) \to \mathbb{R}$$

by

$$\tilde{\omega}(\varphi, \psi) := \int_M \langle G\varphi, \psi \rangle \, \mathrm{dV}. \tag{4.6}$$

Obviously, $\tilde{\omega}$ is $\mathbb{R}$-bilinear and by (4.5) it is skew-symmetric. But $\tilde{\omega}$ does not make $\mathcal{D}(M, E)$ a symplectic vector space because $\tilde{\omega}$ is degenerate. The null space is given by

$$\ker(G) = \{\varphi \in \mathcal{D}(M, E) \mid G\varphi = 0\} = \{\varphi \in \mathcal{D}(M, E) \mid G_+\varphi = G_-\varphi\}.$$

This null space is infinite dimensional because it certainly contains $P(\mathcal{D}(M, E))$ by Theorem 3.4.7. In the globally hyperbolic case this is precisely the null space,

$$\ker(G) = P(\mathcal{D}(M, E)),$$

again by Theorem 3.4.7. On the quotient space $V(M, E, G) := \mathcal{D}(M, E)/\ker(G)$ the degenerate bilinear form $\tilde{\omega}$ induces a symplectic form which we denote by ω.

Lemma 4.3.8. *Let* $X = (M_1, E_1, P_1, G_{1,+}, G_{1,-})$ *and* $Y = (M_2, E_2, P_2, G_{2,+}, G_{2,-})$ *be two objects in* LorFund. *Let* $(f, F) \in \mathrm{Mor}(X, Y)$ *be a morphism.*

Then $\mathrm{ext}: \mathcal{D}(M_1, E_1) \to \mathcal{D}(M_2, E_2)$ *maps the null space* $\ker(G_1)$ *to the null space* $\ker(G_2)$ *and hence induces a symplectic linear map*

$$V(M_1, E_1, G_1) \to V(M_2, E_2, G_2).$$

Proof. If the morphism is the identity, then there is nothing to show. Thus we may assume that M_1 is globally hyperbolic. Let $\varphi \in \ker(G_1)$. Then $\varphi = P_1\psi$ for some $\psi \in \mathcal{D}(M_1, E_1)$ because M_1 is globally hyperbolic. From $G_2(\mathrm{ext}\,\varphi) = G_2(\mathrm{ext}(P_1\psi)) = G_2(P_2(\mathrm{ext}\,\psi)) = 0$ we see that $\mathrm{ext}(\ker(G_1)) \subset \ker(G_2)$. Hence ext induces a linear map $V(M_1, E_1, G_1) \to V(M_2, E_2, G_2)$. From

$$\begin{aligned}\widetilde{\omega}_2(\operatorname{ext}\varphi, \operatorname{ext}\psi) &= \int_{M_2} \langle G_2 \operatorname{ext}\varphi, \operatorname{ext}\psi\rangle \,\mathrm{dV} \\ &= \int_{M_1} \langle \operatorname{res} G_2 \operatorname{ext}\varphi, \psi\rangle \,\mathrm{dV} \\ &= \int_{M_1} \langle G_1\varphi, \psi\rangle \,\mathrm{dV} \\ &= \widetilde{\omega}_1(\varphi, \psi)\end{aligned}$$

we see that this linear map is symplectic. □

We have constructed a functor from the category LorFund to the category SymplVec by mapping each object (M, E, P, G_+, G_-) to $V(M, E, G_+ - G_-)$ and each morphism (f, F) to the symplectic linear map induced by ext. We denote this functor by SYMPL.

We summarize the categories and functors we have defined so far in the following scheme:

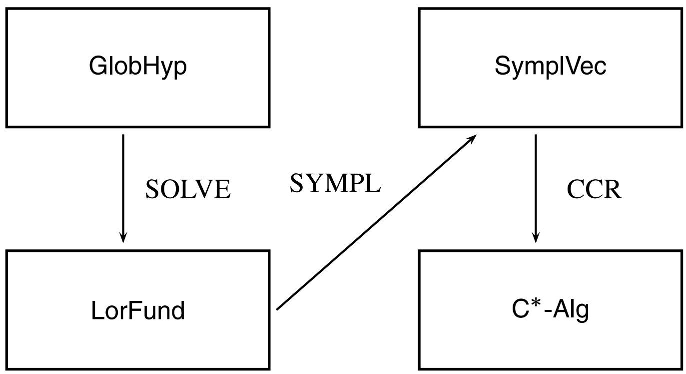


Figure 34. Quantization functors.

4.4 Quasi-local C^*-algebras

The composition of the functors CCR and SYMPL constructed in the previous sections allows us to assign a C^*-algebra to each time-oriented connected Lorentzian manifold equipped with a formally selfadjoint normally hyperbolic operator and Green's operators. Further composing with the functor SOLVE we no longer need to provide Green's operators if we are willing to restrict ourselves to globally hyperbolic manifolds. The elements of this algebra are physically interpreted as the observables related to the field whose wave equation is given by the normally hyperbolic operator.

"Reasonable" open subsets of M are time-oriented Lorentzian manifolds in their own right and come equipped with the restriction of the normally hyperbolic operator over M. Hence each such open subset O yields an algebra whose elements are considered as the observables which can be measured in the spacetime region O. This gives rise to the concept of nets of algebras or quasi-local algebras. A systematic exposition of quasi-local algebras can be found in [Baumgärtel–Wollenberg1992].

Before we define quasi-local algebras we characterize the systems that parametrize the "local algebras". For this we need the notion of directed sets with orthogonality relation.

Definition 4.4.1. A set I is called a *directed set with orthogonality relation* if it carries a partial order $\leq$ and a symmetric relation $\perp$ between its elements such that

(1) for all $\alpha, \beta \in I$ there exists a $\gamma \in I$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$,
(2) for every $\alpha \in I$ there is a $\beta \in I$ with $\alpha \perp \beta$,
(3) if $\alpha \leq \beta$ and $\beta \perp \gamma$, then $\alpha \perp \gamma$,
(4) if $\alpha \perp \beta$ and $\alpha \perp \gamma$, then there exists a $\delta \in I$ such that $\beta \leq \delta$, $\gamma \leq \delta$ and $\alpha \perp \delta$.

In order to handle non-globally hyperbolic manifolds we need to relax this definition slightly:

Definition 4.4.2. A set I is called a *directed set with weak orthogonality relation* if it carries a partial order $\leq$ and a symmetric relation $\perp$ between its elements such that conditions (1), (2), and (3) in Definition 4.4.1 are fulfilled.

Obviously, directed sets with orthogonality relation are automatically directed sets with weak orthogonality relation. We use such sets in the following as index sets for nets of C^*-algebras.

Definition 4.4.3. A (*bosonic*) *quasi-local C^*-algebra* is a pair $(\mathfrak{A}, \{\mathfrak{A}_\alpha\}_{\alpha \in I})$ of a C^*-algebra $\mathfrak{A}$ and a family $\{\mathfrak{A}_\alpha\}_{\alpha \in I}$ of C^*-subalgebras, where I is a directed set with orthogonality relation such that the following holds:

(1) $\mathfrak{A}_\alpha \subset \mathfrak{A}_\beta$ whenever $\alpha \leq \beta$.
(2) $\mathfrak{A} = \overline{\bigcup_\alpha \mathfrak{A}_\alpha}$ where the bar denotes the closure with respect to the norm of $\mathfrak{A}$.
(3) The algebras $\mathfrak{A}_\alpha$ have a common unit 1.
(4) If $\alpha \perp \beta$ the commutator of $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ is trivial: $\left[\mathfrak{A}_\alpha, \mathfrak{A}_\beta\right] = \{0\}$.

Remark 4.4.4. This definition is a special case of the one in [Bratteli–Robinson2002-I, Def. 2.6.3] where there is in addition an involutive automorphism σ of $\mathfrak{A}$. In our case $\sigma = \mathrm{id}$ which physically corresponds to a bosonic theory. This is why one might call our version of quasi-local C^*-algebras *bosonic*.

Definition 4.4.5. A *morphism* between two quasi-local C^*-algebras $(\mathfrak{A}, \{\mathfrak{A}_\alpha\}_{\alpha \in I})$ and $(\mathfrak{B}, \{\mathfrak{B}_\beta\}_{\beta \in J})$ is a pair (φ, Φ) where $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is a unit-preserving C^*-morphism and $\varphi\colon I \to J$ is a map such that:

(1) φ is monotonic, i.e., if $\alpha_1 \leq \alpha_2$ in I then $\varphi(\alpha_1) \leq \varphi(\alpha_2)$ in J,
(2) φ preserves orthogonality, i.e., if $\alpha_1 \perp \alpha_2$ in I, then $\varphi(\alpha_1) \perp \varphi(\alpha_2)$ in J,
(3) $\Phi(\mathfrak{A}_\alpha) \subset \mathfrak{B}_{\varphi(\alpha)}$ for all $\alpha \in I$.

The composition of morphisms of quasi-local C^*-algebras is just the composition of maps, and we obtain the category QuasiLocAlg of quasi-local C^*-algebras.

Definition 4.4.6. A *weak quasi-local C^*-algebra* is a pair $(\mathfrak{A}, \{\mathfrak{A}_\alpha\}_{\alpha\in I})$ of a C^*-algebra $\mathfrak{A}$ and a family $\{\mathfrak{A}_\alpha\}_{\alpha\in I}$ of C^*-subalgebras, where I is a directed set with weak orthogonality relation such that the same conditions as in Definition 4.4.3 hold. Morphisms between weak quasi-local C^*-algebras are defined in exactly the same way as morphisms between quasi-local C^*-algebras.

This yields another category, the category of weak quasi-local C^*-algebras $\mathsf{QuasiLocAlg}_{\mathsf{weak}}$. We note that QuasiLocAlg is a full subcategory of $\mathsf{QuasiLocAlg}_{\mathsf{weak}}$.

Next we want to associate to any object (M, E, P, G_+, G_-) in LorFund a weak quasi-local C^*-algebra. For this we set

$$I := \{O \subset M \mid O \text{ is open, relatively compact, causally compatible, globally hyperbolic } \} \cup \{\emptyset, M\}.$$

On I we take the inclusion $\subset$ as the partial order $\leq$ and define the orthogonality relation by

$$O \perp O' :\Leftrightarrow J^M(\bar{O}) \cap \bar{O}' = \emptyset.$$

This means that two elements of I are orthogonal if and only if they are *causally independent* subsets of M in the sense that there are no causal curves connecting a point in $\bar{O}$ with a point in $\bar{O}'$. Of course, this relation is symmetric.

Lemma 4.4.7. *The set I defined above is a directed set with weak orthogonality relation.*

Proof. Condition (1) in Definition 4.4.1 holds with $\gamma = M$ and (2) with $\beta = \emptyset$. Property (3) is also clear because $O \subset O'$ implies $J^M(\bar{O}) \subset J^M(\bar{O}')$. □

Lemma 4.4.8. *Let M be globally hyperbolic. Then the set I is a directed set with (non-weak) orthogonality relation.*

Proof. In addition to Lemma 4.4.7 we have to show Property (4) of Definition 4.4.1. Let $O_1, O_2, O_3 \in I$ with $J^M(\bar{O}_1) \cap \bar{O}_2 = \emptyset$ and $J^M(\bar{O}_1) \cap \bar{O}_3 = \emptyset$. We want to find $O_4 \in I$ with $O_2 \cup O_3 \subset O_4$ and $J^M(\bar{O}_1) \cap \bar{O}_4 = \emptyset$.

Without loss of generality let O_1, O_2, O_3 be non-empty. Now none of O_1, O_2, and O_3 can equal M. In particular, O_1, O_2, and O_3 are relatively compact. Set $\Omega := M \setminus J^M(\bar{O}_1)$. By Lemma A.5.11 the subset Ω of M is causally compatible and globally hyperbolic. The hypothesis $J^M(\bar{O}_1) \cap \bar{O}_2 = \emptyset = J^M(\bar{O}_1) \cap \bar{O}_3$ implies $\bar{O}_2 \cup \bar{O}_3 \subset \Omega$. Applying Proposition A.5.13 with $K := \bar{O}_2 \cup \bar{O}_3$ in the globally hyperbolic manifold Ω, we obtain a relatively compact causally compatible globally hyperbolic open subset $O_4 \subset \Omega$ containing O_2 and O_3. Since Ω is itself causally compatible in M, the subset O_4 is causally compatible in M as well. By definition of Ω we have $\bar{O}_4 \subset \Omega = M \setminus J^M(\bar{O}_1)$, i.e., $J^M(\bar{O}_1) \cap \bar{O}_4 = \emptyset$. This shows Property (4) and concludes the proof of Lemma 4.4.8. □

Remark 4.4.9. If M is globally hyperbolic, the proof of Proposition A.5.13 shows that the index set I would also be directed if we removed M from it in its definition. Namely, for all elements $O_1, O_2 \in I$ different from $\emptyset$ and M, the element O from Proposition A.5.13 applied to $K := \bar{O}_2 \cup \bar{O}_3$ belongs to I.

Now we are in the situation to associate a weak quasi-local C^*-algebra to any object (M, E, P, G_+, G_-) in LorFund.

We consider the index set I as defined above. For any non-empty $O \in I$ we take the restriction $E|_O$ and the corresponding restriction of the operator P to sections of $E|_O$. Due to the causal compatibility of $O \subset M$ the restrictions of the Green's operators G_+, G_- to sections over O yield the Green's operators G_+^O, G_-^O for P on O, see Proposition 3.5.1. Therefore we get an object $(O, E|_O, P, G_+^O, G_-^O)$ for each $O \in I$, $O \neq \emptyset$.

For $\emptyset \neq O_1 \subset O_2$ the inclusion induces a morphism ι_{O_2,O_1} in the category LorFund. This morphism is given by the embeddings $O_1 \hookrightarrow O_2$ and $E|_{O_1} \hookrightarrow E|_{O_2}$. Let α_{O_2,O_1} denote the morphism $\mathrm{CCR} \circ \mathrm{SYMPL}(\iota_{O_2,O_1})$ in $\mathsf{C^*\text{-}Alg}$. Recall that α_{O_2,O_1} is an injective unit-preserving $*$-morphism.

We set for $\emptyset \neq O \in I$

$$(V_O, \omega_O) := \mathrm{SYMPL}(O, E|_O, P, G_+^O, G_-^O)$$

and for $O \in I$, $O \neq \emptyset$, $O \neq M$,

$$\mathfrak{A}_O := \alpha_{M,O}\left(\mathrm{CCR}(V_O, \omega_O)\right).$$

Obviously, for any $O \in I$, $O \neq \emptyset$, $O \neq M$ the algebra $\mathfrak{A}_O$ is a C^*-subalgebra of $\mathrm{CCR}(V_M, \omega_M)$. For $O = M$ we define $\mathfrak{A}_M$ as the C^*-subalgebra of $\mathrm{CCR}(V_M, \omega_M)$ generated by all $\mathfrak{A}_O$,

$$\mathfrak{A}_M := C^*\Big(\bigcup_{\substack{O \in I \\ O \neq \emptyset, O \neq M}} \mathfrak{A}_O\Big).$$

Finally, for $O = \emptyset$ we set $\mathfrak{A}_\emptyset = \mathbb{C} \cdot 1$. We have thus defined a family $\{\mathfrak{A}_O\}_{O \in I}$ of C^*-subalgebras of $\mathfrak{A}_M$.

Lemma 4.4.10. *Let (M, E, P, G_+, G_-) be an object in* LorFund. *Then $(\mathfrak{A}_M, \{\mathfrak{A}_O\}_{O \in I})$ is a weak quasi-local C^*-algebra.*

Proof. We know from Lemma 4.4.7 that I is a directed set with weak orthogonality relation.

It is clear that $\mathfrak{A}_M = \overline{\bigcup_{O \in I} \mathfrak{A}_O}$ because M belongs to I. By construction it is also clear that all algebras $\mathfrak{A}_O$ have the common unit $W(0)$, $0 \in V_M$. Hence Conditions (2) and (3) in Definition 4.4.3 are obvious.

By functoriality we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{CCR}(V_O,\omega_O) & \xrightarrow{\alpha_{M,O}} & \mathrm{CCR}(V_M,\omega_M) \\ \Big\downarrow \alpha_{O',O} & \nearrow \alpha_{M,O'} & \\ \mathrm{CCR}(V_{O'},\omega_{O'}) \; . & & \end{array}$$

Since $\alpha_{O',O}$ is injective we have $\mathfrak{A}_O \subset \mathfrak{A}_{O'}$. This proves Condition (1) in Definition 4.4.3.

Let now $O, O' \in I$ be causally independent. Let $\varphi \in \mathcal{D}(O, E)$ and $\psi \in \mathcal{D}(O', E)$. It follows from $\mathrm{supp}(G\varphi) \subset J^M(O)$ that $\mathrm{supp}(G\varphi) \cap \mathrm{supp}(\psi) = \emptyset$, hence

$$\int_M \langle G\varphi \, , \psi\rangle \, \mathrm{dV} = 0.$$

For the symplectic form ω on $\mathcal{D}(M, E)/\ker(G)$ this means $\omega(\varphi, \psi) = 0$, where we denote the equivalence class in $\mathcal{D}(M, E)/\ker(G)$ of the extension to M by zero of φ again by φ and similarly for ψ. This yields by Property (iii) of a Weyl-system

$$W(\varphi) \cdot W(\psi) = W(\varphi + \psi) = W(\psi) \cdot W(\varphi),$$

i.e., the generators of $\mathfrak{A}_O$ commute with those of $\mathfrak{A}_{O'}$. Therefore $[\mathfrak{A}_O, \mathfrak{A}_{O'}] = 0$. This proves (4) in Definition 4.4.3. □

Next we associate a morphism in $\mathsf{QuasiLocAlg}_{\mathsf{weak}}$ to any morphism (f, F) in $\mathsf{LorFund}$ between two objects $(M_1, E_1, P_1, G_{1+}, G_{1-})$ and $(M_2, E_2, P_2, G_{2+}, G_{2-})$. Recall that in the case of distinct objects such a morphism only exists if M_1 is globally hyperbolic. Let I_1 and I_2 denote the index sets for the two objects as above and let $\big(\mathfrak{A}_{M_1}, \{\mathfrak{A}_O\}_{O\in I_1}\big)$ and $\big(\mathfrak{B}_{M_2}, \{\mathfrak{B}_O\}_{O\in I_2}\big)$ denote the corresponding weak quasi-local C^*-algebras. Then f maps any $O_1 \in I_1$, $O_1 \neq M_1$, to $f(O_1)$ which is an element of I_2 by definition of $\mathsf{LorFund}$. We get a map $\varphi\colon I_1 \to I_2$ by $M_1 \mapsto M_2$ and $O_1 \mapsto f(O_1)$ if $O_1 \neq M_1$. Since f is an embedding such that $f(M_1) \subset M_2$ is causally compatible, the map φ is monotonic and preserves causal independence.

We now consider the morphism $\Phi = \mathrm{CCR} \circ \mathrm{SYMPL}(f, F)\colon \mathrm{CCR}(V_{M_1}, \omega_{M_1}) \to \mathrm{CCR}(V_{M_2}, \omega_{M_2})$. From the commutative diagram of inclusions and embeddings

$$\begin{array}{ccc} O_1 & \hookrightarrow & O_2 \\ {\scriptstyle f|_{O_1}} \Big\downarrow & & \Big\downarrow {\scriptstyle f|_{O_2}} \\ f(O_1) & \hookrightarrow & M_2 \end{array}$$

we see

$$\begin{aligned} &\Phi(\mathfrak{A}_{O_1}) \\ &\quad = \mathrm{CCR}(\mathrm{SYMPL}(f, F)) \circ \mathrm{CCR}(\mathrm{SYMPL}(\iota_{M,O_1}))(\mathrm{CCR}(V_{O_1}, \omega_{O_1})) \\ &\quad = \mathrm{CCR}(\mathrm{SYMPL}(\iota_{M_2,f(O_1)})) \circ \mathrm{CCR}(\mathrm{SYMPL}(f|_{O_1}, F|_{E|_{O_1}}))(\mathrm{CCR}(V_{O_1}, \omega_{O_1})) \\ &\quad \subset \alpha_{M_2,f(O_1)}(\mathrm{CCR}(V_{f(O_1)}, \omega_{f(O_1)})) \\ &\quad = \mathfrak{B}_{f(O_1)}. \end{aligned}$$

This also implies $\Phi(\mathfrak{A}_{M_1}) \subset \mathfrak{B}_{M_2}$. Therefore the pair $(\varphi, \Phi|_{\mathfrak{A}_{M_1}})$ is a morphism in $\mathsf{QuasiLocAlg}_{\mathsf{weak}}$. We summarize

Theorem 4.4.11. *The assignments* $(M, E, P, G_+, G_-) \mapsto (\mathfrak{A}_M, \{\mathfrak{A}_O\}_{O\in I})$ *and* $(f, F) \mapsto (\varphi, \Phi|_{\mathfrak{A}_{M_1}})$ *yield a functor* $\mathsf{LorFund} \to \mathsf{QuasiLocAlg}_{\mathsf{weak}}$.

Proof. If $f = \mathrm{id}_M$ and $F = \mathrm{id}_E$, then $\varphi = \mathrm{id}_I$ and $\Phi = \mathrm{id}_{\mathfrak{A}_M}$. Similarly, the composition of two morphisms in $\mathsf{LorFund}$ is mapped to the composition of the corresponding two morphisms in $\mathsf{QuasiLocAlg}_{\mathsf{weak}}$. □

Corollary 4.4.12. *The composition of* SOLVE *and the functor from Theorem 4.4.11 yields a functor* $\mathsf{GlobHyp} \to \mathsf{QuasiLocAlg}$. *One gets the following commutative diagram of functors:*

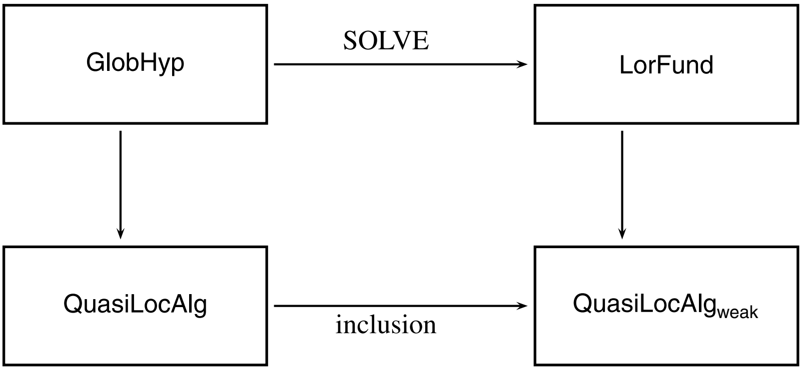


Figure 35. Functors which yield nets of algebras.

Proof. Let (M, E, P) be an object in $\mathsf{GlobHyp}$. Then we know from Lemma 4.4.8 that the index set I associated to $\mathrm{SOLVE}(M, E, P)$ is a directed set with (non-weak) orthogonality relation, and the corresponding weak quasi-local C^*-algebra is in fact a quasi-local C^*-algebra. This concludes the proof since $\mathsf{QuasiLocAlg}$ is a full subcategory of $\mathsf{QuasiLocAlg}_{\mathsf{weak}}$. □

Lemma 4.4.13. *Let* (M, E, P) *be an object in* $\mathsf{GlobHyp}$, *and denote by* $(\mathfrak{A}_M, \{\mathfrak{A}_O\}_{O\in I})$ *the corresponding quasi-local* C^**-algebra. Then*

$$\mathfrak{A}_M = \mathrm{CCR} \circ \mathrm{SYMPL} \circ \mathrm{SOLVE}(M, E, P).$$

Proof. Denote the right-hand side by $\bar{\mathfrak{A}}$. By definition of $\mathfrak{A}_M$ we have $\mathfrak{A}_M \subset \bar{\mathfrak{A}}$.

In order to prove the other inclusion write $(M, E, P, G_+, G_-) := \mathrm{SOLVE}(M, E, P)$. Then $\mathrm{SYMPL}(M, E, P, G_+, G_-)$ is given by $V_M = \mathcal{D}(M, E)/\ker(G)$ with symplectic form ω_M induced by G. Now $\bar{\mathfrak{A}}$ is generated by

$$\mathcal{E} = \{W([\varphi]) \mid \varphi \in \mathcal{D}(M, E)\}$$

where W is the Weyl system from Example 4.2.2 and $[\varphi]$ denotes the equivalence class of φ in V_M. For given $\varphi \in \mathcal{D}(M, E)$ there exists a relatively compact globally hyperbolic causally compatible open subset $O \subset M$ containing the compact set $\mathrm{supp}(\varphi)$ by Proposition A.5.13. For this subset O we have $W([\varphi]) \in \mathfrak{A}_O$. Hence we get $\mathcal{E} \subset \bigcup_{O\in I} \mathfrak{A}_O \subset \mathfrak{A}_M$ which implies $\overline{\mathfrak{A}} \subset \mathfrak{A}_M$. □

Example 4.4.14. Let M be globally hyperbolic. All the operators listed in Examples 4.3.1 to 4.3.3 give rise to quasi-local C^*-algebras. These operators include the d'Alembert operator, the Klein–Gordon operator, the Yamabe operator, the wave operators for the electro-magnetic potential and the Proca field as well as the square of the Dirac operator.

Example 4.4.15. Let M be the anti-deSitter spacetime. Then M is not globally hyperbolic but as we have seen in Section 3.5 we can get Green's operators for the (normalized) Yamabe operator P_g by embedding M conformally into the Einstein cylinder. This yields an object $(M, M \times \mathbb{R}, P_g, G_+, G_-)$ in LorFund. Hence there is a corresponding weak quasi-local C^*-algebra over M.

4.5 Haag–Kastler axioms

We now check that the functor that assigns to each object in LorFund a quasi-local C^*-algebra as constructed in the previous section satisfies the Haag–Kastler axioms of a quantum field theory. These axioms have been proposed in [Haag–Kastler1964, p. 849] for Minkowski space. Dimock [Dimock1980, Sec. 1] adapted them to the case of globally hyperbolic manifolds. He also constructed the quasi-local C^*-algebras for the Klein–Gordon operator.

Theorem 4.5.1. *The functor* LorFund $\to$ QuasiLocAlg$_{\mathrm{weak}}$ *from Theorem 4.4.11 satisfies the Haag–Kastler axioms, that is, for every object (M, E, P, G_+, G_-) in* LorFund *the corresponding weak quasi-local C^*-algebra $(\mathfrak{A}_M, \{\mathfrak{A}_O\}_{O\in I})$ satisfies:*

(1) *If $O_1 \subset O_2$, then $\mathfrak{A}_{O_1} \subset \mathfrak{A}_{O_2}$ for all $O_1, O_2 \in I$.*

(2) $\mathfrak{A}_M = \overline{\bigcup_{O\in I,\ O\neq\emptyset,\ O\neq M} \mathfrak{A}_O}$.

(3) *If M is globally hyperbolic, then $\mathfrak{A}_M$ is simple.*

(4) *The $\mathfrak{A}_O$'s have a common unit* 1.

(5) *For all $O_1, O_2 \in I$ with $J(\overline{O_1}) \cap \overline{O_2} = \emptyset$ the subalgebras $\mathfrak{A}_{O_1}$ and $\mathfrak{A}_{O_2}$ of $\mathfrak{A}_M$ commute: $[\mathfrak{A}_{O_1}, \mathfrak{A}_{O_2}] = \{0\}$.*

(6) *(Time-slice axiom) Let $O_1 \subset O_2$ be nonempty elements of I admitting a common Cauchy hypersurface. Then $\mathfrak{A}_{O_1} = \mathfrak{A}_{O_2}$.*

(7) *Let $O_1, O_2 \in I$ and let the Cauchy development $D(O_2)$ be relatively compact in M. If $O_1 \subset D(O_2)$, then $\mathfrak{A}_{O_1} \subset \mathfrak{A}_{O_2}$.*

Remark 4.5.2. It can happen that the Cauchy development $D(O)$ of a causally compatible globally hyperbolic subset O in a globally hyperbolic manifold M is not relatively compact even if O itself is relatively compact. See the following picture for an example where M and O are "lens-like" globally hyperbolic subsets of Minkowski space:

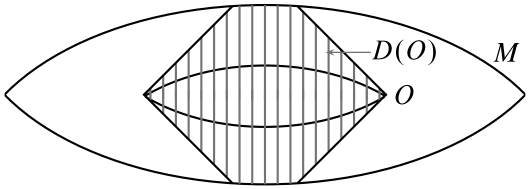


Figure 36. Cauchy development $D(O)$ is *not* relatively compact in M.

This is why we *assume* in (7) that $D(O_2)$ is relatively compact.

Remark 4.5.3. Instead of (3) one often finds the requirement that $\mathfrak{A}_M$ should be *primitive* for globally hyperbolic M. This means that there exists a faithful irreducible representation of $\mathfrak{A}_M$ on a Hilbert space. We know by Lemma 4.4.13 and Corollary 4.2.10 that $\mathfrak{A}_M$ is simple, i.e., that (3) holds. Simplicity implies primitivity because each C^*-algebra has irreducible representations [Bratteli–Robinson2002-I, Sec. 2.3.4].

Proof of Theorem 4.5.1. Only axioms (6) and (7) require a proof. First note that axiom (7) follows from axioms (1) and (6):

Let $O_1, O_2 \in I$, let the Cauchy development $D(O_2)$ be relatively compact in M, and let $O_1 \subset D(O_2)$. By Theorem 1.3.10 there is a smooth spacelike Cauchy hypersurface $\Sigma \subset O_2$. It follows from the definitions that $D(O_2) = D(\Sigma)$. Since O_2 is causally compatible in M the hypersurface Σ is acausal in M. By Lemma A.5.9 $D(\Sigma)$ is a causally compatible and globally hyperbolic open subset of M. Since $D(O_2) = D(\Sigma)$ is relatively compact by assumption we have $D(O_2) \in I$.

Axiom (6) implies $\mathfrak{A}_{O_2} = \mathfrak{A}_{D(O_2)}$. By axiom (1) $\mathfrak{A}_{O_1} \subset \mathfrak{A}_{D(O_2)} = \mathfrak{A}_{O_2}$.

It remains to show the time-slice axiom. We prepare the proof by first deriving two lemmas. The first lemma is of technical nature while the second one is essentially equivalent to the time-slice axiom.

Lemma 4.5.4. *Let O be a causally compatible globally hyperbolic open subset of a globally hyperbolic manifold M. Assume that there exists a Cauchy hypersurface Σ of O which is also a Cauchy hypersurface of M. Let h be a Cauchy time-function on O as in Corollary* 1.3.12 *(applied to O). Let $K \subset M$ be compact. Assume that there exists a $t \in \mathbb{R}$ with $K \subset I_+^M(h^{-1}(t))$.*

Then there is a smooth function $\rho\colon M \to [0,1]$ such that

- *$\rho = 1$ on a neighborhood of K,*
- *$\mathrm{supp}(\rho) \cap J_-^M(K) \subset M$ is compact, and*
- *$\overline{\{x \in M \mid 0 < \rho(x) < 1\}} \cap J_-^M(K)$ is compact and contained in O.*

Remark 4.5.5. Similarly, if instead of $K \subset I_+^M(h^{-1}(t))$ we have $K \subset I_-^M(h^{-1}(t))$ for some t, then we can find a smooth function $\rho\colon M \to [0,1]$ such that

- $\rho = 1$ on a neighborhood of K,
- $\mathrm{supp}(\rho) \cap J_+^M(K) \subset M$ is compact, and
- $\overline{\{x \in M \mid 0 < \rho(x) < 1\}} \cap J_+^M(K)$ is compact and contained in O.

Proof of Lemma 4.5.4. By assumption there exist real numbers $t_- < t_+$ in the range of h such that $K \subset I_+^M(S_{t_+})$, hence also $K \subset I_+^M(S_{t_-})$, where $S_t := h^{-1}(t)$. Since S_t is a Cauchy hypersurface of O and since O and M admit a common Cauchy hypersurface, it follows from Lemma A.5.10 that S_{t_-} and S_{t_+} are also Cauchy hypersurfaces of M. Since $J_+^M(S_{t_+})$ and $J_-^M(S_{t_-})$ are disjoint closed subsets of M there exists a smooth function $\rho\colon M \to [0,1]$ such that $\rho|_{J_+^M(S_{t_+})} = 1$ and $\rho|_{J_-^M(S_{t_-})} = 0$.

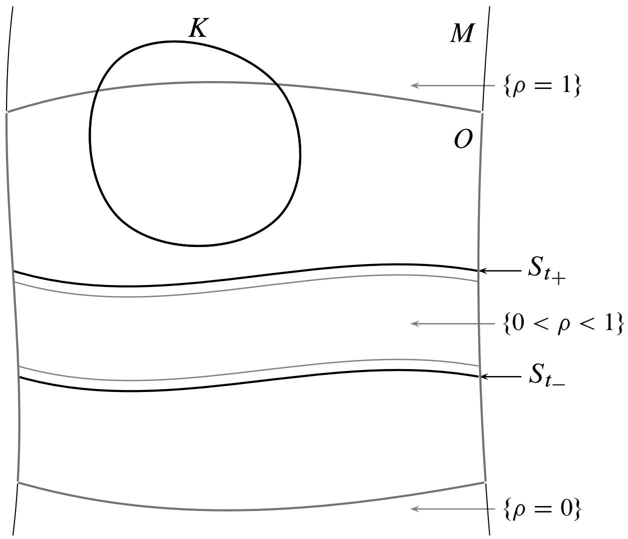


Figure 37. Construction of ρ.

We check that ρ has the three properties stated in Lemma 4.5.4. The first one follows from $K \subset I_+^M(S_{t_+})$.

Since $\rho|_{I_-^M(S_{t_-})} = 0$, we have $\mathrm{supp}(\rho) \subset J_+^M(S_{t_-})$. It follows from Lemma A.5.3 applied to the past-compact subset $J_+^M(S_{t_-})$ of M that $J_+^M(S_{t_-}) \cap J_-^M(K)$ is relatively compact, hence compact by Lemma A.5.1. Therefore the second property holds.

The closed set $\overline{\{0 < \rho < 1\}} \cap J_-^M(K)$ is contained in the compact set $\mathrm{supp}(\rho) \cap J_-^M(K)$, hence compact itself.

The subset $\overline{\{0 < \rho < 1\}}$ of M lies in $J_+^M(S_{t_-}) \cap J_-^M(S_{t_+})$. We claim that $J_+^M(S_{t_-}) \cap J_-^M(S_{t_+}) \subset O$ which will then imply $\overline{\{0 < \rho < 1\}} \cap J_-^M(K) \subset O$ and hence conclude the proof.

Assume that there exists $p \in J_+^M(S_{t_-}) \cap J_-^M(S_{t_+})$ but $p \notin O$. Choose a future directed causal curve $c\colon [s_-, s_+] \to M$ from S_{t_-} to S_{t_+} through p. Extend this curve to an inextendible future directed causal curve $c\colon \mathbb{R} \to M$. Let I' be the connected component of $c^{-1}(O)$ containing s_-. Then $I' \subset \mathbb{R}$ is an open interval and $c|_{I'}$ is an inextendible causal curve in O. Since $p \notin O$ the curve leaves O before it reaches S_{t_+}, hence $s_+ \notin I'$. But S_{t_+} is a Cauchy hypersurface in O and so there must be an $s \in I'$ with $c(s) \in S_{t_+}$. Therefore the curve c meets S_{t_+} at least twice (namely in s and in s_+) in contradiction to S_{t_+} being a Cauchy hypersurface in M. □

Lemma 4.5.6. *Let (M, E, P) be an object in* GlobHyp *and let O be a causally compatible globally hyperbolic open subset of M. Assume that there exists a Cauchy hypersurface Σ of O which is also a Cauchy hypersurface of M. Let $\varphi \in \mathcal{D}(M, E)$.*

Then there exist $\psi, \chi \in \mathcal{D}(M, E)$ such that $\mathrm{supp}(\psi) \subset O$ *and*

$$\varphi = \psi + P\chi.$$

Proof of Lemma 4.5.6. Let h be a time-function on O as in Corollary 1.3.12 (applied to O). Fix $t_- < t_+ \in \mathbb{R}$ in the range of h. By Lemma A.5.10 the subsets $S_{t_-} := h^{-1}(t_-)$ and $S_{t_+} := h^{-1}(t_+)$ are also Cauchy hypersurfaces of M. Hence every inextendible timelike curve in M meets S_{t_-} and S_{t_+}. Since $t_- < t_+$ the set $\{I_+^M(S_{t_-}), I_-^M(S_{t_+})\}$ is a finite open cover of M.

Let $\{f_+, f_-\}$ be a partition of unity subordinated to this cover. In particular, $\mathrm{supp}(f_\pm) \subset I_\pm^M(S_{t_\mp})$. Set $K_\pm := \mathrm{supp}(f_\pm\varphi) = \mathrm{supp}(\varphi) \cap \mathrm{supp}(f_\pm)$. Then $K_\pm$ is a compact subset of M satisfying $K_\pm \subset I_\pm^M(S_{t_\mp})$. Applying Lemma 4.5.4 we obtain two smooth functions $\rho_+, \rho_-\colon M \to [0, 1]$ satisfying

- $\rho_\pm = 1$ in a neighborhood of $K_\pm$,
- $\mathrm{supp}(\rho_\pm) \cap J_\mp^M(K_\pm) \subset M$ is compact, and
- $\overline{\{0 < \rho_\pm < 1\}} \cap J_\mp^M(K_\pm)$ is compact and contained in O.

Set $\chi_\pm := \rho_\pm G_\mp(f_\pm\varphi)$, $\chi := \chi_+ + \chi_-$ and $\psi := \varphi - P\chi$. By definition, $\chi_\pm$, χ, and ψ are smooth sections in E over M. Since $\mathrm{supp}(G_\mp(f_\pm\varphi)) \subset J_\mp^M(\mathrm{supp}(f_\pm\varphi)) \subset J_\mp^M(K_\pm)$, the support of $\chi_\pm$ is contained in $\mathrm{supp}(\rho_\pm) \cap J_\mp^M(K_\pm)$, which is compact by the second property of $\rho_\pm$. Therefore $\chi \in \mathcal{D}(M, E)$.

It remains to show that $\mathrm{supp}(\psi)$ is compact and contained in O. By the first property of $\rho_\pm$ one has $\chi_\pm = G_\mp(f_\pm\varphi)$ in a neighborhood of $K_\pm$. Moreover, $f_\pm\varphi = 0$ on $\{\rho_\pm = 0\}$. Hence $P\chi_\pm = f_\pm\varphi$ on $\{\rho_\pm = 0\} \cup \{\rho_\pm = 1\}$. Therefore $f_\pm\varphi - P\chi_\pm$ vanishes outside $\overline{\{0 < \rho_\pm < 1\}}$, i.e., $\mathrm{supp}(f_\pm\varphi - P\chi_\pm) \subset \overline{\{0 < \rho_\pm < 1\}}$. By the definitions of $\chi_\pm$ and $f_\pm$ one also has $\mathrm{supp}(f_\pm\varphi - P\chi_\pm) \subset K_\pm \cup J_\mp^M(K_\pm) = J_\mp^M(K_\pm)$, hence $\mathrm{supp}(f_\pm\varphi - P\chi_\pm) \subset \overline{\{0 < \rho_\pm < 1\}} \cap J_\mp^M(K_\pm)$, which is compact and contained in O by the third property of $\rho_\pm$. Therefore the support of $\psi = f_+\varphi - P\chi_+ + f_-\varphi - P\chi_-$ is compact and contained in O. This shows Lemma 4.5.6. □

End of proof of Theorem 4.5.1. The time-slice axiom in Theorem 4.5.1 follows directly from Lemma 4.5.6. Namely, let $O_1 \subset O_2$ be nonempty causally compatible globally hyperbolic open subsets of M admitting a common Cauchy hypersurface. Let $[\varphi] \in V_{O_2} := \mathcal{D}(O_2, E)/\ker(G_{O_2})$. Lemma 4.5.6 applied to $M := O_2$ and $O := O_1$ yields $\chi \in \mathcal{D}(O_2, E)$ and $\psi \in \mathcal{D}(O_1, E)$ such that $\varphi = \text{ext}\,\psi + P\chi$. Since $P\chi \in \ker(G_{O_2})$ we have $[\varphi] = [\text{ext}\,\psi]$, that is, $[\varphi]$ is the image of the symplectic linear map $V_{O_1} \to V_{O_2}$ induced by the inclusion $O_1 \hookrightarrow O_2$, compare Lemma 4.3.8. We see that this symplectic map is surjective, hence an isomorphism. Symplectic isomorphisms induce isomorphisms of C^*-algebras, hence the inclusion $\mathfrak{A}_{O_1} \subset \mathfrak{A}_{O_2}$ is actually an equality. This proves the time-slice axiom and concludes the proof of Theorem 4.5.1. □

4.6 Fock space

In quantum mechanics a particle is described by its wave function which mathematically is a solution u to an equation $Pu = 0$. We consider normally hyperbolic operators P in this text. The passage from single particle systems to multi particle systems is known as *second quantization* in the physics literature. Mathematically it requires the construction of the quantum field which we will do in the subsequent section. In this section we will describe some functional analytical underpinnings, namely the construction of the bosonic Fock space.

We start by describing the symmetric tensor product of Hilbert spaces. Let H denote a complex vector space. We will use the convention that the Hermitian scalar product $(\cdot,\cdot)$ on H is anti-linear in the first argument. Let H_n be the vector space freely generated by $H \times \cdots \times H$ (n copies), i.e., the space of all finite formal linear combinations of elements of $H \times \cdots \times H$. Let V_n be the vector subspace of H_n generated by all elements of the form $(v_1, \dots, cv_k, \dots, v_n) - c\cdot(v_1, \dots, v_k, \dots, v_n)$, $(v_1, \dots, v_k + v_k', \dots, v_n) - (v_1, \dots, v_k, \dots, v_n) - (v_1, \dots, v_k', \dots, v_n)$ and $(v_1, \dots, v_n) - (v_{\sigma(1)}, \dots, v_{\sigma(n)})$ where $v_j, v_j' \in H$, $c \in \mathbb{C}$ and σ a permutation.

Definition 4.6.1. The vector space $\bigodot_{\text{alg}}^n H := H_n/V_n$ is called the *algebraic n-th symmetric tensor product* of H. By convention, we put $\bigodot_{\text{alg}}^0 H := \mathbb{C}$.

For the equivalence class of $(v_1, \dots, v_n) \in H_n$ in $\bigodot_{\text{alg}}^n H$ we write $v_1 \odot \cdots \odot v_n$. The map $\gamma\colon H \times \cdots \times H \to \bigodot_{\text{alg}}^n H$ given by $\gamma(v_1, \dots, v_n) = v_1 \odot \cdots \odot v_n$ is multilinear and symmetric. The algebraic symmetric tensor product has the following universal property.

Lemma 4.6.2. *For each complex vector space W and each symmetric multilinear map $\alpha\colon H \times \cdots \times H \to W$ there exists one and only one linear map $\bar\alpha\colon \bigodot_{\text{alg}}^n H \to W$*

such that the diagram

$$\begin{array}{ccc} H \times \cdots \times H & & \\ \downarrow \gamma & \overset{\alpha}{\searrow} & \\ \bigodot_{\text{alg}}^{n} H & \xrightarrow{\bar{\alpha}} & W \end{array}$$

commutes.

Proof. Uniqueness of $\bar{\alpha}$ is clear because

$$\bar{\alpha}(v_1 \odot \cdots \odot v_n) = \alpha(v_1, \ldots, v_n) \tag{4.7}$$

and the elements $v_1 \odot \cdots \odot v_n$ generate $\bigodot_{\text{alg}}^{n} H$.

To show existence one defines $\bar{\alpha}$ by Equation 4.7 and checks easily that this is well defined. □

The algebraic symmetric tensor product $\bigodot_{\text{alg}}^{n} H$ inherits a scalar product from H characterized by

$$(v_1 \odot \cdots \odot v_n, w_1 \odot \cdots \odot w_n) = \sum_{\sigma} (v_1, w_{\sigma(1)}) \ldots (v_n, w_{\sigma(n)})$$

where the sum is taken over all permutations σ on $\{1, \ldots, n\}$.

Definition 4.6.3. The completion of $\bigodot_{\text{alg}}^{n} H$ with respect to this scalar product is called the *n-th symmetric tensor product* of the Hilbert space H and is denoted $\bigodot^{n} H$. In particular, $\bigodot^{0} H = \mathbb{C}$.

Remark 4.6.4. If $\{e_j\}_{j \in \mathcal{J}}$ is an orthonormal system of H where $\mathcal{J}$ is some ordered index set, then $\{e_{j_1} \odot \cdots \odot e_{j_n}\}_{j_1 \leq \cdots \leq j_n}$ forms an orthogonal system of $\bigodot^{n} H$. For each ordered multiindex $J = (j_1, \ldots, j_n)$ there is a corresponding partition of n, $n = k_1 + \cdots + k_l$, given by

$$j_1 = \cdots = j_{k_1} < j_{k_1+1} = \cdots = j_{k_1+k_2} < \cdots < j_{k_1+\cdots+k_{l-1}+1} = \cdots = j_n.$$

We compute

$$\begin{aligned} \|e_{j_1} \odot \cdots \odot e_{j_n}\|^2 &= \sum_{\sigma} (e_{j_1}, e_{j_{\sigma(1)}}) \ldots (e_{j_n}, e_{j_{\sigma(n)}}) \\ &= \#\{\sigma \mid (j_{\sigma(1)}, \ldots, j_{\sigma(n)}) = (j_1, \ldots, j_n)\} \\ &= k_1! \ldots k_l!. \end{aligned}$$

In particular,

$$1 \leq \|e_{j_1} \odot \cdots \odot e_{j_n}\| \leq \sqrt{n!}.$$

The algebraic direct sum $\mathcal{F}_{\text{alg}}(H) := \bigoplus_{\text{alg},n=0}^{\infty} \bigodot^n H$ carries a natural scalar product, namely

$$((w_0, w_1, w_2, \dots), (u_0, u_1, u_2, \dots)) = \sum_{n=0}^{\infty} (w_n, u_n)$$

where $w_n, u_n \in \bigodot^n H$.

Definition 4.6.5. We call $\mathcal{F}_{\text{alg}}(H)$ the *algebraic symmetric Fock space* of H. The completion of $\mathcal{F}_{\text{alg}}(H)$ with respect to this scalar product is denoted $\mathcal{F}(H)$ and is called the *bosonic* or *symmetric Fock space* of H. The vector $\Omega := 1 \in \mathbb{C} = \bigodot^0 H \subset \mathcal{F}_{\text{alg}}(H) \subset \mathcal{F}(H)$ is called the *vacuum vector*.

The elements of the Hilbert space $\mathcal{F}(H)$ are therefore sequences $(w_0, w_1, w_2, \dots)$ with $w_n \in \bigodot^n H$ such that

$$\sum_{n=0}^{\infty} \|w_n\|^2 < \infty.$$

Fix $v \in H$. The map $\alpha \colon H \times \dots \times H \to \bigodot^{n+1} H$, $\alpha(v_1, \dots, v_n) = v \odot v_1 \odot \dots \odot v_n$, is symmetric multilinear and induces a linear map $\bar{\alpha} \colon \bigodot_{\text{alg}}^n H \to \bigodot^{n+1} H$, $v_1 \odot \dots \odot v_n \mapsto v \odot v_1 \odot \dots \odot v_n$, by Lemma 4.6.2. We compute the operator norm of $\bar{\alpha}$. Without loss of generality we can assume $\|v\| = 1$. We choose the orthonormal system $\{e_j\}_{j \in \mathcal{J}}$ of H such that v belongs to it. If v is perpendicular to all $e_{j_1}, \dots, e_{j_n}$, then

$$\|\bar{\alpha}(e_{j_1} \odot \dots \odot e_{j_n})\| = \|v \odot e_{j_1} \odot \dots \odot e_{j_n}\| = \|e_{j_1} \odot \dots \odot e_{j_n}\|.$$

If v is one of the e_{j_μ}, say $v = e_{j_1}$, then

$$\begin{aligned} \|\bar{\alpha}(e_{j_1} \odot \dots \odot e_{j_n})\|^2 &= (k_1 + 1)!\, k_2! \dots k_l! \\ &= (k_1 + 1)\|e_{j_1} \odot \dots \odot e_{j_n}\|^2. \end{aligned}$$

Thus in any case

$$\|\bar{\alpha}(e_{j_1} \odot \dots \odot e_{j_n})\| \le \sqrt{n+1}\|e_{j_1} \odot \dots \odot e_{j_n}\|$$

and equality holds for $e_{j_1} = \dots = e_{j_n} = v$. Dropping the assumption $\|v\| = 1$ this shows

$$\|\bar{\alpha}\| = \sqrt{n+1}\|v\|.$$

Hence $\bar{\alpha}$ extends to a bounded linear map

$$a^*(v) \colon \bigodot^n H \to \bigodot^{n+1} H, \quad a^*(v)(v_1 \odot \dots \odot v_n) = v \odot v_1 \odot \dots \odot v_n$$

with

$$\|a^*(v)\| = \sqrt{n+1}\|v\|. \tag{4.8}$$

For the vacuum vector this means $a^*(v)\Omega = v$. The map $a^*(v)$ is naturally defined as a linear map $\mathcal{F}_{\mathrm{alg}}(H) \to \mathcal{F}_{\mathrm{alg}}(H)$. By (4.8) $a^*(v)$ is unbounded on $\mathcal{F}_{\mathrm{alg}}(H)$ unless $v = 0$ and therefore does not extend continuously to $\mathcal{F}(H)$. Writing $v = v_0$ we see

$$\begin{aligned}
&(a^*(v)(v_1 \odot \cdots \odot v_n), w_0 \odot w_1 \odot \cdots \odot w_n) \\
&\quad = (v_0 \odot v_1 \odot \cdots \odot v_n, w_0 \odot w_1 \odot \cdots \odot w_n) \\
&\quad = \sum_\sigma (v_0, w_{\sigma(0)})(v_1, w_{\sigma(1)}) \ldots (v_n, w_{\sigma(n)}) \\
&\quad = \sum_{k=0}^{n} \sum_{\substack{\sigma \text{ with} \\ \sigma(0)=k}} (v, w_k)(v_1, w_{\sigma(1)}) \ldots (v_n, w_{\sigma(n)}) \\
&\quad = \Big(v_1 \odot \cdots \odot v_n, \sum_{k=0}^{n} (v, w_k) w_0 \odot \cdots \odot \hat{w}_k \odot \cdots \odot w_n \Big)
\end{aligned}$$

where $\hat{w}_k$ indicates that the factor w_k is left out. Hence if we define $a(v)\colon \bigodot_{\mathrm{alg}}^{n+1} H \to \bigodot_{\mathrm{alg}}^{n} H$ by

$$a(v)(w_0 \odot \cdots \odot w_n) = \sum_{k=0}^{n} (v, w_k) w_0 \odot \cdots \odot \hat{w}_k \odot \cdots \odot w_n$$

and $a(v)\Omega = 0$, then we have

$$(a^*(v)\omega, \eta) = (\omega, a(v)\eta) \tag{4.9}$$

for all $\omega \in \bigodot_{\mathrm{alg}}^{n} H$ and $\eta \in \bigodot_{\mathrm{alg}}^{n+1} H$. The operator norm of $a(v)$ is easily determined:

$$\begin{aligned}
\|a(v)\| &= \sup_{\substack{\eta \in \bigodot_{\mathrm{alg}}^{n+1} H \\ \|\eta\|=1}} \|a(v)\eta\| = \sup_{\substack{\eta \in \bigodot_{\mathrm{alg}}^{n+1} H \\ \omega \in \bigodot_{\mathrm{alg}}^{n} H \\ \|\eta\|=\|\omega\|=1}} (\omega, a(v)\eta) \\
&= \sup_{\substack{\eta \in \bigodot_{\mathrm{alg}}^{n+1} H \\ \omega \in \bigodot_{\mathrm{alg}}^{n} H \\ \|\eta\|=\|\omega\|=1}} (a^*(v)\omega, \eta) = \sup_{\substack{\omega \in \bigodot_{\mathrm{alg}}^{n} H \\ \|\omega\|=1}} \|a^*(v)\omega\| \\
&= \|a^*(v)\| = \sqrt{n+1}\, \|v\|.
\end{aligned}$$

Thus $a(v)$ extends continuously to a linear operator $a(v)\colon \bigodot^{n+1} H \to \bigodot^{n} H$ with

$$\|a(v)\| = \sqrt{n+1}\, \|v\|. \tag{4.10}$$

We consider both $a^*(v)$ and $a(v)$ as unbounded linear operators in Fock space $\mathcal{F}(H)$ with $\mathcal{F}_{\mathrm{alg}}(H)$ as invariant domain of definition. In the physics literature, $a(v)$ is known as *annihilation operator* and $a^*(v)$ as *creation operator* for $v \in H$.

Lemma 4.6.6. *Let H be a complex Hilbert space and let $v, w \in H$. Then the canonical commutator relations (CCR) hold, i.e.,*

$$[a(v), a(w)] = [a^*(v), a^*(w)] = 0,$$

$$[a(v), a^*(w)] = (v, w)\mathrm{id}.$$

Proof. From

$$\begin{aligned} a^*(v)a^*(w)v_1 \odot \cdots \odot v_n &= v \odot w \odot v_1 \odot \cdots \odot v_n \\ &= w \odot v \odot v_1 \odot \cdots \odot v_n \\ &= a^*(w)a^*(v)v_1 \odot \cdots \odot v_n \end{aligned}$$

we see directly that $[a^*(v), a^*(w)] = 0$. By (4.9) we have for all $\omega, \eta \in \mathcal{F}_{\mathrm{alg}}(H)$

$$(\omega, [a(v), a(w)]\eta) = ([a^*(w), a^*(v)]\omega, \eta) = 0.$$

Since $\mathcal{F}_{\mathrm{alg}}(H)$ is dense in $\mathcal{F}(H)$ this implies $[a(v), a(w)]\eta = 0$. Finally, subtracting

$$\begin{aligned} a(v)a^*(w)v_1 \odot \cdots \odot v_n &= a(v)w \odot v_1 \odot \cdots \odot v_n \\ &= (v, w)v_1 \odot \cdots \odot v_n \\ &\quad + \sum_{k=1}^{n} (v, v_k) w \odot v_1 \odot \cdots \odot \hat{v}_k \odot \cdots \odot v_n \end{aligned}$$

and

$$\begin{aligned} a^*(w)a(v)v_1 \odot \cdots \odot v_n &= w \odot \sum_{k=1}^{n} (v, v_k) v_1 \odot \cdots \odot \hat{v}_k \odot \cdots \odot v_n \\ &= \sum_{k=1}^{n} (v, v_k) w \odot v_1 \odot \cdots \odot \hat{v}_k \odot \cdots \odot v_n \end{aligned}$$

yields $[a(v), a^*(w)](v_1 \odot \cdots \odot v_n) = (v, w)v_1 \odot \cdots \odot v_n$. □

Definition 4.6.7. Let H be a complex Hilbert space and let $v \in H$. We define the *Segal field* as the unbounded operator

$$\theta(v) := \frac{1}{\sqrt{2}}(a(v) + a^*(v))$$

in $\mathcal{F}(H)$ with $\mathcal{F}_{\mathrm{alg}}(H)$ as domain of definition.

Notice that $a^*(v)$ depends $\mathbb{C}$-linearly on v while $v \mapsto a(v)$ is anti-linear. Hence $\theta(v)$ is only $\mathbb{R}$-linear in v.

Lemma 4.6.8. *Let H be a complex Hilbert space and let $v \in H$. Then the Segal operator $\theta(v)$ is essentially selfadjoint.*

Proof. Since $\theta(v)$ is symmetric by (4.9) and densely defined it is closable. The domain of definition $\mathcal{F}_{\mathrm{alg}}(H)$ is invariant for $\theta(v)$ and hence all powers $\theta(v)^m$ are defined on it. By Nelson's theorem is suffices to show that all vectors in $\mathcal{F}_{\mathrm{alg}}(H)$ are analytic, see Theorem A.2.18. Since all vectors in the domain of definition are finite linear combinations of vectors in $\bigodot^n H$ for various n we only need to show that $\omega \in \bigodot^n H$ is analytic. By (4.8), (4.10), and the fact that $a^*(v)\omega \in \bigodot^{n+1} H$ and $a(v)\omega \in \bigodot^{n-1} H$ are perpendicular we have

$$\begin{aligned}\|\theta(v)\omega\|^2 &= \frac{1}{2}\|a^*(v)\omega + a(v)\omega\|^2 \\ &= \frac{1}{2}(\|a^*(v)\omega\|^2 + \|a(v)\omega\|^2) \\ &\leq \frac{1}{2}((n+1)\|v\|^2\|\omega\|^2 + n\|v\|^2\|\omega\|^2) \\ &\leq (n+1)\|v\|^2\|\omega\|^2,\end{aligned}$$

hence

$$\|\theta(v)^m\omega\| \leq \sqrt{(n+1)(n+2)\dots(n+m)}\|v\|^m\|\omega\|. \tag{4.11}$$

For any $t > 0$

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{t^m}{m!}\|\theta(v)^m\omega\| &\leq \sum_{m=0}^{\infty} \frac{t^m}{m!}\sqrt{(n+1)(n+2)\dots(n+m)}\,\|v\|^m\|\omega\| \\ &= \sum_{m=0}^{\infty} \frac{t^m}{\sqrt{m!}}\sqrt{\frac{n+1}{1}\cdot\frac{n+2}{2}\dots\frac{n+m}{m}}\,\|v\|^m\|\omega\| \\ &\leq \sum_{m=0}^{\infty} \frac{t^m}{\sqrt{m!}}\sqrt{n+1}^m\,\|v\|^m\|\omega\| \\ &< \infty\end{aligned}$$

because the power series $\sum_{m=0}^{\infty} \frac{x^m}{\sqrt{m!}}$ has infinite radius of convergence. Thus ω is an analytic vector. □

Lemma 4.6.9. *Let H be a complex Hilbert space and let $v, v_j, w \in H$, $j = 1, 2, \dots$ Let $\eta \in \mathcal{F}_{\mathrm{alg}}(H)$. Then the following holds:*

(1) $(\theta(v)\theta(w) - \theta(w)\theta(v))\eta = i\,\mathfrak{Im}(v, w)\eta$.

(2) *If* $\|v - v_j\| \to 0$, *then* $\|\theta(v)\eta - \theta(v_j)\eta\| \to 0$ *as* $j \to \infty$.

(3) *The linear span of the vectors* $\theta(v_1)\dots\theta(v_n)\Omega$ *where* $v_j \in H$ *and* $n \in \mathbb{N}$ *is dense in* $\mathcal{F}(H)$.

Proof. We see (1) by Lemma 4.6.6

$$\begin{aligned}\theta(v)\theta(w)\eta &= \frac{1}{2}(a^*(v)+a(v))(a^*(w)+a(w))\eta \\ &= \frac{1}{2}(a^*(v)a^*(w)+a^*(v)a(w)+a(v)a^*(w)+a(v)a(w))\eta \\ &= \frac{1}{2}(a^*(w)a^*(v)+a(w)a^*(v)-(w,v)\mathrm{id}+a(w)a^*(w)+(v,w)\mathrm{id} \\ &\quad +a(w)a(v))\eta \\ &= \theta(w)\theta(v)\eta + i\mathfrak{Im}(v,w)\eta.\end{aligned}$$

For (2) it suffices to prove the statement for $\eta \in \bigodot^n H$. By (4.8) and (4.10) we see

$$\begin{aligned}\|\theta(v)\eta-\theta(v_j)\eta\| &= 2^{-1/2}\|a^*(v)\eta+a(v)\eta-a^*(v_j)\eta-a(v_j)\eta\| \\ &\le 2^{-1/2}(\|a^*(v)\eta-a^*(v_j)\eta\|+\|a(v)\eta-a(v_j)\eta\|) \\ &= 2^{-1/2}(\|a^*(v-v_j)\eta\|+\|a(v-v_j)\eta\|) \\ &\le 2^{-1/2}(\sqrt{n+1}+\sqrt{n})\|v-v_j\|\|\eta\|\end{aligned}$$

which implies the statement.

For (3) one can easily see by induction on N that the span of the vectors $\theta(v_1)\dots\theta(v_n)\Omega$, $n \le N$, and the span of the vectors $a^*(v_1)\dots a^*(v_n)\Omega$, $n \le N$, both coincide with $\bigoplus_{n=0}^N \bigodot_{\mathrm{alg}}^n H$. This is dense in $\bigoplus_{n=0}^N \bigodot^n H$ and the assertion follows. □

Now we can relate this discussion to the canonical commutator relations as studied in Section 4.2. Denote the (selfadjoint) closure of $\theta(v)$ again by $\theta(v)$ and denote the domain of this closure by $\mathrm{dom}(\theta(v))$. Look at the unitary operator $W(v) := \exp(i\theta(v))$. Recall that for the analytic vectors $\omega \in \mathcal{F}_{\mathrm{alg}}(H)$ we have the series

$$W(v)\omega = \sum_{m=0}^{\infty} \frac{i^m}{m!}\theta(v)^m\omega$$

converging absolutely.

Proposition 4.6.10. *For $v, v_j, w \in H$ we have the following:*

(1) *The domain* $\mathrm{dom}(\theta(w))$ *is preserved by* $W(v)$, *i.e.,* $W(v)(\mathrm{dom}(\theta(w))) = \mathrm{dom}(\theta(w))$ *and*

$$W(v)\theta(w)\omega = \theta(w)W(v)\omega - \mathfrak{Im}(v,w)W(v)\omega$$

for all $\omega \in \mathrm{dom}(\theta(w))$.

(2) *The map* $W: H \to \mathcal{L}(\mathcal{F}(H))$ *is a Weyl system of the symplectic vector space* $(H, \mathfrak{Im}(\cdot,\cdot))$.

(3) *If* $\|v-v_j\| \to 0$, *then* $\|(W(v)-W(v_j))\eta\| \to 0$ *for all* $\eta \in \mathcal{F}(H)$.

Proof. We first check the formula (1) for $\omega \in \mathcal{F}_{\mathrm{alg}}(H)$. From Lemma 4.6.9 (1) we get inductively

$$\theta(v)^m\theta(w)\omega = \theta(w)\theta(v)^m\omega + i\cdot m\cdot \mathfrak{Im}(v,w)\,\theta(v)^{m-1}\omega.$$

Since $\theta(w)\omega \in \mathcal{F}_{\mathrm{alg}}(H)$ we have

$$\begin{aligned}
\theta(w)W(v)\omega &= \sum_{m=0}^{\infty}\frac{i^m}{m!}\theta(w)\theta(v)^m\omega\\
&= \sum_{m=0}^{\infty}\frac{i^m}{m!}\left(\theta(v)^m\theta(w) - im\;\mathfrak{Im}(v,w)\,\theta(v)^{m-1}\right)\omega\\
&= W(v)\theta(w)\omega - \sum_{m=1}^{\infty}\frac{i^{m+1}\mathfrak{Im}(v,w)}{(m-1)!}\,\theta(v)^{m-1}\omega\\
&= W(v)\theta(w)\omega + \mathfrak{Im}(v,w)W(v)\omega. \qquad (4.12)
\end{aligned}$$

In particular, $W(v)\omega$ is an analytic vector for $\theta(w)$ and we have for $\omega,\eta \in \mathcal{F}_{\mathrm{alg}}(H)$

$$\begin{aligned}
\|\theta(w)W(v)(\omega-\eta)\| &= \|(W(v)\theta(w) + \mathfrak{Im}(v,w)W(v))(\omega-\eta)\|\\
&\le \|W(v)\theta(w)(\omega-\eta)\| + |\mathfrak{Im}(v,w)|\|W(v)(\omega-\eta)\|\\
&\le (\|\theta(w)(\omega-\eta)\| + |\mathfrak{Im}(v,w)|\cdot\|\omega-\eta\|)\cdot\|v\|. \qquad (4.13)
\end{aligned}$$

Now let $\omega \in \mathrm{dom}(\theta(w))$. Then there exist $\omega_j \in \mathcal{F}_{\mathrm{alg}}(H)$ such that $\|\omega-\omega_j\| \to 0$ and $\|\theta(w)(\omega) - \theta(w)(\omega_j)\| \to 0$ as $j \to \infty$. Since $W(v)$ is bounded we have $\|W(v)(\omega) - W(v)(\omega_j)\| \to 0$ and by (4.13) $\{\theta(w)W(v)(\omega_j)\}_j$ is a Cauchy sequence and therefore convergent as well. Hence $W(v)(\omega) \in \mathrm{dom}(\theta(w))$ and the validity of (4.12) extends to all $\omega \in \mathrm{dom}(\theta(w))$.

We have shown $W(v)(\mathrm{dom}(\theta(w))) \subset \mathrm{dom}(\theta(w))$. Replacing $W(v)$ by $W(-v) = W(v)^{-1}$ yields $W(v)(\mathrm{dom}(\theta(w))) = \mathrm{dom}(\theta(w))$.

For (2) observe that $W(0) = \exp(0) = \mathrm{id}$ and $W(-v) = \exp(i\theta(-v)) = \exp(-i\theta(v)) = \exp(i\theta(v))^* = W(v)^*$. We fix $\omega \in \mathcal{F}_{\mathrm{alg}}(H)$ and look at the smooth curve

$$x(t) := W(tv)W(tw)W(-t(v+w))\omega$$

in $\mathcal{F}(H)$. We have $x(0) = \omega$ and

$$\begin{aligned}
\dot{x}(t) &= \frac{d}{dt}W(tv)W(tw)W(-t(v+w))\omega\\
&= iW(tv)\theta(v)W(tw)W(-t(v+w))\omega + iW(tv)W(tw)\theta(w)W(-t(v+w))\omega\\
&\quad + iW(tv)W(tw)\theta(-v-w)W(-t(v+w))\omega\\
&= iW(tv)\theta(v)W(tw)W(-t(v+w))\omega + iW(tv)W(tw)\theta(-v)W(-t(v+w))\omega\\
&\overset{(1)}{=} iW(tv)W(tw)(\theta(v) + \mathfrak{Im}(tw,v))W(-t(v+w))\omega\\
&\quad + iW(tv)W(tw)\theta(-v)W(-t(v+w))\omega\\
&= i\cdot t\cdot\mathfrak{Im}(w,v)\cdot x(t).
\end{aligned}$$

Therefore $x(t) = e^{i\mathfrak{Im}(w,v)t^2/2}\omega$, and $t = 1$ yields $W(v)W(w)W(-(v+w))\omega = e^{i\mathfrak{Im}(w,v)/2}\omega$. By continuity this equation extends to all $\omega \in \mathcal{F}(H)$ and shows that W is a Weyl system for the symplectic form $\mathfrak{Im}(\cdot,\cdot)$.

For (3) let $\eta \in \bigodot^n H$ and let $\|v - v_j\| \to 0$ as $j \to \infty$. Then

$$\begin{aligned}
\|(W(v) - W(v_j))\eta\| &= \|W(v)(\mathrm{id} - W(-v)W(v_j))\eta\| \\
&\leq \|(\mathrm{id} - W(-v)W(v_j))\eta\| \\
&\leq \|(1 - e^{i\mathfrak{Im}(v,v_j)/2})\eta\| + \|(e^{i\mathfrak{Im}(v,v_j)/2} - W(-v)W(v_j))\eta\| \\
&\overset{(2)}{=} \|(1 - e^{i\mathfrak{Im}(v,v_j)/2})\eta\| \\
&\quad + \|(e^{i\mathfrak{Im}(v,v_j)/2} - e^{i\mathfrak{Im}(v,v_j)/2}W(v_j - v))\eta\| \\
&\leq |1 - e^{i\mathfrak{Im}(v,v_j)/2}| \cdot \|\eta\| + \|(\mathrm{id} - W(v_j - v))\eta\|.
\end{aligned}$$

Since $\mathfrak{Im}(v, v_j) = \frac{1}{2}\mathfrak{Im}(v - v_j, v + v_j) \to 0$ it suffices to show $\|(\mathrm{id} - W(v_j - v))\eta\| \to 0$. This follows from

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{1}{m!}\|\theta(v_j - v)^m \eta\| &\overset{(4.11)}{\leq} \sum_{m=1}^{\infty} \sqrt{\frac{(n+1)^m}{m!}}\|v_j - v\|^m \|\eta\| \\
&= \|v_j - v\| \sum_{m=0}^{\infty} \sqrt{\frac{(n+1)^{m+1}}{(m+1)!}}\|v_j - v\|^m \|\eta\|
\end{aligned}$$

and $\sum_{m=0}^{\infty} \sqrt{\frac{(n+1)^{m+1}}{(m+1)!}}\|v_j - v\|^m < \infty$ uniformly in j. We have seen that $\|(W(v) - W(v_j))\eta\| \to 0$ for all $\eta \in \bigodot^n H$ (n fixed) hence for all $\eta \in \mathcal{F}_{\mathrm{alg}}(H)$.

Finally, let $\eta \in \mathcal{F}(H)$ be arbitrary. Let $\varepsilon > 0$. Choose $\eta' \in \mathcal{F}_{\mathrm{alg}}(H)$ such that $\|\eta - \eta'\| < \varepsilon$. For $j \gg 0$ we have $\|(W(v) - W(v_j))\eta'\| < \varepsilon$. Hence

$$\begin{aligned}
\|(W(v) - W(v_j))\eta\| &\leq \|(W(v) - W(v_j))(\eta - \eta')\| + \|(W(v) - W(v_j))\eta'\| \\
&\leq 2\|\eta - \eta'\| + \varepsilon \\
&< 3\varepsilon.
\end{aligned}$$

This concludes the proof. □

4.7 The quantum field defined by a Cauchy hypersurface

In this final section we construct the quantum field. This yields a formulation of the quantized theory on Fock space which is closer to the traditional presentations of quantum field theory than the formulation in terms of quasi-local C^*-algebras given in Sections 4.4 and 4.5. It has the disadvantage however of depending on a choice of Cauchy hypersurface. Even worse from a physical point of view, this quantum field has all the properties that one usually requires except for one, the "microlocal spectrum condition". We do not discuss this condition in the present book, see the remarks at the end of this section and the references mentioned therein. The construction given here is nevertheless useful because it illustrates how the abstract algebraic formulation of quantum field theory relates to more traditional ones.

Let (M, E, P) be an object in the category GlobHyp, i.e., M is a globally hyperbolic Lorentzian manifold, E is a real vector bundle over M with nondegenerate inner product, and P is a formally selfadjoint normally hyperbolic operator acting on sections in E. We need an additional piece of structure.

Definition 4.7.1. Let $k \in \mathbb{N}$. A *twist structure of spin* $k/2$ on E is a smooth section $Q \in C^\infty(M, \mathrm{Hom}(\bigodot^k TM, \mathrm{End}(E)))$ with the following properties:

(1) Q is symmetric with respect to the inner product on E, i.e.,

$$\langle Q(X_1 \odot \cdots \odot X_k)e, f\rangle = \langle e, Q(X_1 \odot \cdots \odot X_k)f\rangle$$

for all $X_j \in T_pM$, $e, f \in E_p$, and $p \in M$.

(2) If X is future directed timelike, then the bilinear form $\langle\cdot,\cdot\rangle_X$ defined by

$$\langle f, g\rangle_X := \langle Q(X \odot \cdots \odot X)f, g\rangle$$

is positive definite.

Note that the bilinear form $\langle\cdot,\cdot\rangle_X$ is symmetric by (1) so that (2) makes sense. From now on we write $Q_X := Q(X \odot \cdots \odot X)$ for brevity. If X is past directed timelike, then $\langle\cdot,\cdot\rangle_X$ is positive or negative definite depending on the parity of k. Note furthermore, that Q_X is a field of *isomorphisms* of E in case that X is timelike since otherwise $\langle\cdot,\cdot\rangle_X$ would be degenerate.

Examples 4.7.2. a) Let E be a real vector bundle over M with Riemannian metric. In the case of the d'Alembert, the Klein–Gordon, and the Yamabe operator we are in this situation. We take $k = 0$ and $Q : \bigodot^0 TM = \mathbb{R} \to \mathrm{End}(E)$, $t \mapsto t \cdot \mathrm{id}$. By convention, $Q_X = \mathrm{id}$ and hence $\langle\cdot,\cdot\rangle_X = \langle\cdot,\cdot\rangle$ for a timelike vector X of unit length.

b) Let M carry a spin structure and let $E = \Sigma M$ be the spinor bundle. The Dirac operator and its square act on sections in ΣM. As explained in [Baum1981, Sec. 3.3] and [Bär–Gauduchon–Moroianu2005, Sec. 2] there is a natural indefinite Hermitian product $(\cdot,\cdot)$ on ΣM such that for future directed timelike X the sesquilinear form $(\cdot,\cdot)_X$ defined by

$$(\varphi, \psi)_X = (\varphi, X \cdot \psi)$$

is symmetric and positive definite where "$\cdot$" denotes Clifford multiplication. Hence if we view ΣM as a real bundle and put $\langle\cdot,\cdot\rangle := \Re e(\cdot,\cdot)$, $k := 1$, and $Q(X)\varphi := X \cdot \varphi$, then we have a twist structure of spin $1/2$ on the spinor bundle.

c) On the bundle of p-forms $E = \Lambda^p T^*M$ there is a natural indefinite inner product $\langle\cdot,\cdot\rangle$ characterized by

$$\langle \alpha, \beta\rangle = \sum_{0 \le i_1 < \cdots < i_p \le n} \varepsilon_{i_1} \ldots \varepsilon_{i_p} \cdot \alpha(e_{i_1}, \ldots, e_{i_p}) \cdot \beta(e_{i_1}, \ldots, e_{i_p})$$

where $e_1, \dots, e_n$ is an orthonormal basis with $\varepsilon_i = \langle e_i, e_i \rangle = \pm 1$. We put $k := 2$ and

$$Q(X \odot Y)\alpha := X^\flat \wedge \iota_Y \alpha + Y^\flat \wedge \iota_X \alpha - \langle X, Y \rangle \cdot \alpha$$

where ι_X denotes insertion of X in the first argument, $\iota_X \alpha = \alpha(X, \cdot, \dots, \cdot)$, and $X \mapsto X^\flat$ is the natural isomorphism $TM \to T^*M$ induced by the Lorentzian metric. It is easy to check that Q is a twist structure of spin 1 on $\Lambda^p T^*M$. Recall that the case $p = 1$ is relevant for the wave equation in electrodynamics and for the Proca equation.

The physically oriented reader will have noticed that in all these examples $k/2$ indeed coincides with the spin of the particle under consideration.

Remark 4.7.3. If Q is a twist structure on E, then Q^* is a twist structure on E^* of the same spin where $Q^*(X_1 \odot \cdots \odot X_k) = Q(X_1 \odot \cdots \odot X_k)^*$ is given by the adjoint map. On E^*, we will always use this induced twist structure without further comment.

Let us return to the construction of the quantum field for the object (M, E, P) in GlobHyp. As an additional data we fix a twist structure Q on E. As usual let E^* denote the dual bundle and P^* the adjoint operator acting on sections in E^*. Let $G^*_\pm$ denote the Green's operators for P^* and $G^* = G^*_+ - G^*_-$.

We choose a spacelike smooth Cauchy hypersurface $\Sigma \subset M$. We denote by $L^2(\Sigma, E^*)$ the real Hilbert space of square integrable sections in E^* over Σ with scalar product

$$(u, v)_\Sigma := \int_\Sigma \langle u, v \rangle_\mathfrak{n} \, \mathrm{dA} = \int_\Sigma \langle Q^*_\mathfrak{n} u, v \rangle \, \mathrm{dA}$$

where $\mathfrak{n}$ denotes the future directed (timelike) unit normal to Σ. Here $\langle \cdot, \cdot \rangle$ denotes the inner product on E^* inherited from the one on E. Let $H_\Sigma := L^2(\Sigma, E^*) \otimes_\mathbb{R} \mathbb{C}$ be the complexification of this real Hilbert space and extend $(\cdot, \cdot)_\Sigma$ to a Hermitian scalar product on H_Σ thus turning H_Σ into a complex Hilbert space. We use the convention that $(\cdot, \cdot)_\Sigma$ is conjugate linear in the first argument.

We construct the symmetric Fock space $\mathcal{F}(H_\Sigma)$ as in the previous section. Let θ be the corresponding Segal field.

Given $f \in \mathcal{D}(M, E^*)$ the smooth section $G^* f$ is contained in $C^\infty_{\mathrm{sc}}(M, E^*)$, i.e., there exists a compact subset $K \subset M$ such that $\mathrm{supp}(G^* f) \subset J^M(K)$, see Theorem 3.4.7. It thus follows from Corollary A.5.4 that the intersection $\mathrm{supp}(G^* f) \cap \Sigma$ is compact and $G^* f|_\Sigma \in \mathcal{D}(\Sigma, E^*) \subset L^2(\Sigma, E^*) \subset H_\Sigma$. Similarly, $\nabla_\mathfrak{n}(G^* f) \in \mathcal{D}(\Sigma, E^*) \subset L^2(\Sigma, E^*) \subset H_\Sigma$. We can therefore define

$$\Phi_\Sigma(f) := \theta(i(G^* f)|_\Sigma - (Q^*_\mathfrak{n})^{-1} \nabla_\mathfrak{n}(G^* f)).$$

Definition 4.7.4. The map Φ_Σ from $\mathcal{D}(M, E^*)$ to the set of selfadjoint operators on Fock space $\mathcal{F}(H_\Sigma)$ is called the *quantum field* (or the *field operator*) for P defined by Σ.

Notice that Φ_Σ depends upon the choice of the Cauchy hypersurface Σ. One thinks of Φ_Σ as an operator-valued distribution on M. This can be made more precise.

Proposition 4.7.5. *Let (M, E, P) be an object in the category* GlobHyp *with a twist structure Q. Choose a spacelike smooth Cauchy hypersurface $\Sigma \subset M$. Let Φ_Σ be the quantum field for P defined by Σ.*

Then for every $\omega \in \mathcal{F}_{\mathrm{alg}}(H_\Sigma)$ the map

$$\mathcal{D}(M, E^*) \to \mathcal{F}(H_\Sigma), \quad f \mapsto \Phi_\Sigma(f)\omega,$$

is continuous. In particular, the map

$$\mathcal{D}(M, E^*) \to \mathbb{C}, \quad f \mapsto (\eta, \Phi_\Sigma(f)\omega),$$

is a distributional section in E for any $\eta, \omega \in \mathcal{F}_{\mathrm{alg}}(H_\Sigma)$.

Proof. Let $f_j \to f$ in $\mathcal{D}(M, E^*)$. Then $G^* f_j \to G^* f$ in $C^\infty_{\mathrm{sc}}(M, E^*)$ by Proposition 3.4.8. Thus $G^* f_j|_\Sigma \to G^* f|_\Sigma$ and $(Q^*_{\mathfrak{n}})^{-1}\nabla_{\mathfrak{n}} G^* f_j \to (Q^*_{\mathfrak{n}})^{-1}\nabla_{\mathfrak{n}} G^* f$ in $\mathcal{D}(\Sigma, E^*)$. Hence $G^* f_j|_\Sigma \to G^* f|_\Sigma$ and $(Q^*_{\mathfrak{n}})^{-1}\nabla_{\mathfrak{n}} G^* f_j \to (Q^*_{\mathfrak{n}})^{-1}\nabla_{\mathfrak{n}} G^* f$ in H_Σ. The proposition now follows from Lemma 4.6.9 (2). □

The quantum field satisfies the equation $P\Phi_\Sigma = 0$ in the distributional sense. More precisely, we have

Proposition 4.7.6. *Let (M, E, P) be an object in the category* GlobHyp *with a twist structure Q. Choose a spacelike smooth Cauchy hypersurface $\Sigma \subset M$. Let Φ_Σ be the quantum field for P defined by Σ.*

For every $f \in \mathcal{D}(M, E^*)$ one has

$$\Phi_\Sigma(P^* f) = 0.$$

Proof. This is clear from $G^* P^* f = 0$ and $\theta(0) = 0$. □

To proceed we need the following reformulation of Lemma 3.2.2.

Lemma 4.7.7. *Let (M, E, P) be an object in the category* GlobHyp*, let $G_\pm$ be the Green's operators for P and let $G = G_+ - G_-$. Furthermore, let $\Sigma \subset M$ be a spacelike Cauchy hypersurface with future directed (timelike) unit normal vector field $\mathfrak{n}$.*

Then for all $f, g \in \mathcal{D}(M, E)$,

$$\int_M \langle f, Gg \rangle \, \mathrm{dV} = \int_\Sigma (\langle \nabla_{\mathfrak{n}}(Gf), Gg \rangle - \langle Gf, \nabla_{\mathfrak{n}}(Gg) \rangle) \, \mathrm{dA}\,.$$

Proof. Since $J^M_+(\Sigma)$ is past compact and $J^M_-(\Sigma)$ is future compact Lemma 3.2.2 applies. After identification of E^* with E via the inner product $\langle \cdot , \cdot \rangle$ the assertion follows from Lemma 3.2.2 with $u = Gg$. □

The quantum field satisfies the following commutator relation.

Proposition 4.7.8. *Let (M, E, P) be an object in the category* GlobHyp *with a twist structure Q. Choose a spacelike smooth Cauchy hypersurface $\Sigma \subset M$. Let Φ_Σ be the quantum field for P defined by Σ.*

Then for all $f, g \in \mathcal{D}(M, E^*)$ and all $\eta \in \mathcal{F}_{\mathrm{alg}}(H_\Sigma)$ one has

$$[\Phi_\Sigma(f), \Phi_\Sigma(g)]\eta = i \cdot \int_M \langle G^* f, g\rangle \,\mathrm{dV} \cdot \eta.$$

Proof. Using Lemma 4.6.9 and the fact that $(\cdot,\cdot)_\Sigma$ is the complexification of a real scalar product we compute

$$\begin{aligned}
&[\Phi_\Sigma(f), \Phi_\Sigma(g)]\eta \\
&\quad= \left[\theta\left(i(G^* f)|_\Sigma - (Q_{\mathfrak{n}}^*)^{-1}\nabla_{\mathfrak{n}}(G^* f)\right), \theta\left(i(G^* g)|_\Sigma - (Q_{\mathfrak{n}}^*)^{-1}\nabla_{\mathfrak{n}}(G^* g)\right)\right]\eta \\
&\quad= i\,\mathfrak{Im}\left(i(G^* f)|_\Sigma - (Q_{\mathfrak{n}}^*)^{-1}\nabla_{\mathfrak{n}}(G^* f), i(G^* g)|_\Sigma - (Q_{\mathfrak{n}}^*)^{-1}\nabla_{\mathfrak{n}}(G^* g)\right)_\Sigma \eta \\
&\quad= -i\,\mathfrak{Im}\left(i(G^* f)|_\Sigma, (Q_{\mathfrak{n}}^*)^{-1}\nabla_{\mathfrak{n}}(G^* g)\right)_\Sigma \eta \\
&\qquad\quad - i\,\mathfrak{Im}\left((Q_{\mathfrak{n}}^*)^{-1}\nabla_{\mathfrak{n}}(G^* f), i(G^* g)|_\Sigma\right)_\Sigma \eta \\
&\quad= i\left((G^* f)|_\Sigma, (Q_{\mathfrak{n}}^*)^{-1}\nabla_{\mathfrak{n}}(G^* g)\right)_\Sigma \cdot \eta - i\left((Q_{\mathfrak{n}}^*)^{-1}\nabla_{\mathfrak{n}}(G^* f), (G^* g)|_\Sigma\right)_\Sigma \cdot \eta \\
&\quad= i \cdot \int_\Sigma \langle (G^* f)|_\Sigma, \nabla_{\mathfrak{n}}(G^* g)\rangle \,\mathrm{dV} \cdot \eta - i \cdot \int_\Sigma \langle \nabla_{\mathfrak{n}}(G^* f), (G^* g)|_\Sigma\rangle \,\mathrm{dV} \cdot \eta.
\end{aligned}$$

Lemma 4.7.7 applied to P^* concludes the proof. □

Corollary 4.7.9. *Let (M, E, P) be an object in the category* GlobHyp *with a twist structure Q. Choose a spacelike smooth Cauchy hypersurface $\Sigma \subset M$. Let Φ_Σ be the quantum field for P defined by Σ. If the supports of f and $g \in \mathcal{D}(M, E^*)$ are causally independent, then*

$$[\Phi_\Sigma(f), \Phi_\Sigma(g)] = 0.$$

Proof. If the supports of $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are causally independent, then $\mathrm{supp}(G^* f) \subset J^M(\mathrm{supp}(f))$ and $\mathrm{supp}(g)$ are disjoint. Hence

$$[\Phi_\Sigma(f), \Phi_\Sigma(g)] = i \cdot \int_M \langle G^* f, g\rangle \,\mathrm{dV} = 0. \qquad \square$$

Proposition 4.7.10. *Let (M, E, P) be an object in the category* GlobHyp *with a twist structure Q. Choose a spacelike smooth Cauchy hypersurface $\Sigma \subset M$. Let Φ_Σ be the quantum field for P defined by Σ. Let Ω be the vacuum vector in $\mathcal{F}(H_\Sigma)$.*

Then the linear span of the vectors $\Phi_\Sigma(f_1)\dots\Phi_\Sigma(f_n)\Omega$ is dense in $\mathcal{F}(H_\Sigma)$ where $f_j \in \mathcal{D}(M, E^*)$ and $n \in \mathbb{N}$.

Proof. By Lemma 4.6.9 (3) the span of vectors of the form $\theta(v_1)\dots\theta(v_n)\Omega$, $v_j \in H_\Sigma$, $n \in \mathbb{N}$, is dense in $\mathcal{F}(H_\Sigma)$. It therefore suffices to approximate vectors of the form $\theta(v_1)\dots\theta(v_n)\Omega$ by vectors of the form $\Phi_\Sigma(f_1)\dots\Phi_\Sigma(f_n)\Omega$. Any $v_j \in H_\Sigma$ is of the form $v_j = w_j + iz_j$ with $w_j, z_j \in L^2(\Sigma, E^*)$. Since $\mathcal{D}(\Sigma, E^*)$ is dense in

$L^2(\Sigma, E^*)$ we may assume without loss of generality that $w_j, z_j \in \mathcal{D}(\Sigma, E^*)$ by Proposition 4.7.5.

By Theorem 3.2.11 there exists a solution $u_j \in C^\infty_{\mathrm{sc}}(M, E^*)$ to the Cauchy problem $Pu_j = 0$ with initial conditions $u_j|_\Sigma = z_j$ and $\nabla_\mathfrak{n} u_j = -Q^*_\mathfrak{n} w_j$. By Theorem 3.4.7 there exists $f_j \in \mathcal{D}(M, E^*)$ with $G^* f_j = u_j$. It now follows that $\Phi_\Sigma(f_j) = \theta(-(Q^*_\mathfrak{n})^{-1}\nabla_\mathfrak{n}(G^* f_j) + i(G^* f_j)|_\Sigma) = \theta(-(Q^*_\mathfrak{n})^{-1}\nabla_\mathfrak{n}(u_j) + iu_j|_\Sigma) = \theta(w_j + iz_j) = \theta(v_j)$. This concludes the proof. □

Remark 4.7.11. In the physics literature one usually also finds that the quantum field should satisfy

$$\Phi_\Sigma(\bar{f}) = \Phi_\Sigma(f)^*. \tag{4.14}$$

This simply expresses the fact that we are dealing with a real theory and that the quantum field takes its values in self-adjoint operators. Recall that we have assumed E to be a *real* vector bundle. Of course, one could complexify E and extend Φ_Σ complex linearly such that (4.14) holds.

We relate the quantum field constructed in this section to the CCR-algebras studied earlier.

Proposition 4.7.12. *Let (M, E, P) be an object in the category* GlobHyp *and let Q be a twist structure on E^*. Choose a spacelike smooth Cauchy hypersurface $\Sigma \subset M$. Let Φ_Σ be the quantum field for P defined by Σ.*

Then the map

$$W_\Sigma \colon \mathcal{D}(M, E^*) \to \mathcal{L}(\mathcal{F}(H_\Sigma)), \quad W_\Sigma(f) = \exp(i\,\Phi_\Sigma(f)),$$

yields a Weyl system of the symplectic vector space $\mathrm{SYMPL} \circ \mathrm{SOLVE}(M, E^*, P^*)$.

Proof. Recall that the symplectic vector space $\mathrm{SYMPL} \circ \mathrm{SOLVE}(M, E^*, P^*)$ is given by $V(M, E^*, G^*) = \mathcal{D}(M, E^*)/\ker(G^*)$ with symplectic form induced by $\tilde{\omega}(f, g) = \int_M \langle G^* f, g\rangle\, \mathrm{dV}$. By definition $W_\Sigma(f) = 1$ holds for any $f \in \ker(G^*)$, hence W_Σ descends to a map $V(M, E^*, G^*) \to \mathcal{L}(\mathcal{F}(H_\Sigma))$.

Let $f, g \in \mathcal{D}(M, E^*)$. Set $u := i(G^* f)|_\Sigma - (Q^*_\mathfrak{n})^{-1}\nabla_\mathfrak{n}(G^* f)$ and $v := i(G^* g)|_\Sigma - (Q^*_\mathfrak{n})^{-1}\nabla_\mathfrak{n}(G^* g) \in H_\Sigma$ so that $\Phi_\Sigma(f) = \theta(u)$ and $\Phi_\Sigma(g) = \theta(v)$. Then by Lemma 4.6.9 (1) and by Proposition 4.7.8 we have

$$i\,\mathfrak{Im}(u, v)_\Sigma \cdot \mathrm{id} = [\theta(u), \theta(v)] = [\Phi_\Sigma(f), \Phi_\Sigma(g)] = i \int_M \langle G^* f, g\rangle\, \mathrm{dV} \cdot \mathrm{id},$$

hence

$$\mathfrak{Im}(u, v)_\Sigma = \int_M \langle G^* f, g\rangle\, \mathrm{dV} = \tilde{\omega}(f, g).$$

Now the result follows from Proposition 4.6.10 (2). □

Corollary 4.7.13. *Let (M, E, P) be an object in the category* GlobHyp *and let Q be a twist structure on E^*. Choose a spacelike smooth Cauchy hypersurface $\Sigma \subset M$. Let W_Σ be the Weyl system defined by Φ_Σ.*

Then the CCR-algebra generated by the $W_\Sigma(f)$, $f \in \mathcal{D}(M, E^*)$, *is isomorphic to* $\mathrm{CCR}(\mathrm{SYMPL}(\mathrm{SOLVE}(M, E^*, P^*)))$.

Proof. This is a direct consequence of Proposition 4.7.12 and of Theorem 4.2.9. □

The construction of the quantum field on a globally hyperbolic Lorentzian manifold goes back to [Isham1978], [Hajicek1978], [Dimock1980], and others in the case of scalar fields, i.e., if E is the trivial line bundle. See also the references in [Fulling1989] and [Wald1994]. In [Dimock1980] the formula $W_\Sigma(f) = \exp(i\,\Phi_\Sigma(f))$ in Proposition 4.7.12 was used to *define* the CCR-algebra. It should be noted that this way one does not get a true quantization functor $\mathsf{GlobHyp} \to \mathsf{C^*\text{-}Alg}$ because one determines the C^*-algebra *up to isomorphism only*. This is caused by the fact that there is no canonical choice of Cauchy hypersurface. It seems that the approach based on algebras of observables as developed in Sections 4.3 to 4.5 is more natural in the context of curved spacetimes than the more traditional approach via the Fock space.

Wave equations for sections in nontrivial vector bundles also appear frequently. The approach presented in this book works for linear wave equations in general but often extra problems have to be taken care of. In [Dimock1992] the electromagnetic field is studied. Here one has to take the gauge freedom into account. For the Proca equation as studied e.g. in [Furlani1999] the extra constraint $\delta A = 0$ must be considered, compare Example 4.3.2. If one wants to study the Dirac equation itself rather than its square as we did in Example 4.3.3, then one has to use the canonical anticommutator relations (CAR) instead of the CCR, see e.g. [Dimock1982].

In the physics papers mentioned above the authors fix a wave equation, e.g. the Klein–Gordon equation, and then they set up a functor $\mathsf{GlobHyp}_{\mathsf{naked}} \to \mathsf{C^*\text{-}Alg}$. Here $\mathsf{GlobHyp}_{\mathsf{naked}}$ is the category whose objects are globally hyperbolic Lorentzian manifolds without any further structure and the morphisms are the time-orientation preserving isometric embeddings $f : M_1 \to M_2$ such that $f(M_1)$ is a causally compatible open subset of M_2. The relation to our more universal functor $\mathrm{CCR} \circ \mathrm{SYMPL} \circ \mathrm{SOLVE} : \mathsf{GlobHyp} \to \mathsf{C^*\text{-}Alg}$ is as follows:

There is the forgetful functor $\mathrm{FORGET} : \mathsf{GlobHyp} \to \mathsf{GlobHyp}_{\mathsf{naked}}$ given by $\mathrm{FORGET}(M, E, P) = M$ and $\mathrm{FORGET}(f, F) = f$. A *geometric normally hyperbolic operator* is a functor $\mathrm{GOp} : \mathsf{GlobHyp}_{\mathsf{naked}} \to \mathsf{GlobHyp}$ with $\mathrm{FORGET} \circ \mathrm{GOp} = \mathrm{id}$.

For example, the Klein–Gordon equation for fixed mass m yields such a functor. One puts $\mathrm{GOp}(M) := (M, E, P)$ where E is the trivial real line bundle over M with the canonical inner product and P is the Klein–Gordon operator $P = \Box + m^2$. On the level of morphisms, one sets $\mathrm{GOp}(f) := (f, F)$ where F is the embedding $M_1 \times \mathbb{R} \hookrightarrow M_2 \times \mathbb{R}$ induced by $f : M_1 \hookrightarrow M_2$. Similarly, the Yamabe operator, the wave equations for the electromagnetic field and for the Proca field yield geometric normally hyperbolic operators.

The square of the Dirac operator does not yield a geometric normally hyperbolic operator because the construction of the spinor bundle depends on the additional choice of a spin structure. One can of course fix this by incorporating the spin structure into yet another category, the category of globally hyperbolic Lorentzian manifolds equipped with a spin structure, see [Verch2001, Sec. 3].

In any case, given a geometric normally hyperbolic operator GOp, then $\mathrm{CCR} \circ \mathrm{SYMPL} \circ \mathrm{SOLVE} \circ \mathrm{GOp} \colon \mathsf{GlobHyp}_{\mathsf{naked}} \to \mathsf{C^*\text{-}Alg}$ is a *locally covariant quantum field theory* in the sense of [Brunetti–Fredenhagen–Verch2003, Def. 2.1].

For introductions to quantum field theory on curved spacetimes from the physical point of view the reader is referred to the books [Birrell–Davies1984], [Fulling1989], and [Wald1994].

The passage from the abstract quantization procedure yielding quasi-local C^*-algebras to the more familiar concept based on Fock space and quantum fields requires certain choices (Cauchy hypersurface) and additional structures (twist structure) and is therefore not canonical. Furthermore, there are many more Hilbert space representations than the Fock space representations constructed here and the question arises which ones are physically relevant. A criterion in terms of micro-local analysis was found in [Radzikowski1996]. As a matter of fact, the Fock space representations constructed here turn out not to satisfy this criterion and are therefore nowadays regarded as unphysical. A good geometric understanding of the physical Hilbert space representations on a general globally hyperbolic spacetime is still missing.

Radzikowski's work was developed further in [Brunetti–Fredenhagen–Köhler1996] and applied in [Brunetti–Fredenhagen1997] to interacting fields. The theory of interacting quantum fields, in particular their renormalizability, currently forms an area of very active research.

Appendix. Background material

In Sections A.1 to A.4 the necessary terminology and basic facts from such diverse fields of mathematics as category theory, functional analysis, differential geometry, and differential operators are presented. These sections are included for the convenience of the reader and are not meant to be a substitute for a thorough introduction to these topics.

Section A.5 is of a different nature. Here we collect advanced material on Lorentzian geometry which is needed in the main text. In this section we give full proofs. Partly due to the technical nature of many of these results they have not been included in the main text in order not to distract the reader.

A.1 Categories

We start with basic definitions and examples from category theory (compare [Lang2002, Ch. 1, § 11]). A nice introduction to further concepts related to categories can be found in [MacLane1998].

Definition A.1.1. A *category* A consists of the following data:

- a class $\mathrm{Obj}(\mathsf{A})$ whose members are called *objects*
- for any two objects $A, B \in \mathrm{Obj}(\mathsf{A})$ there is a (possibly empty) set $\mathrm{Mor}(A, B)$ whose elements are called *morphisms*,
- for any three objects $A, B, C \in \mathrm{Obj}(\mathsf{A})$ there is a map (called the *composition* of morphisms)

$$\mathrm{Mor}(B, C) \times \mathrm{Mor}(A, B) \to \mathrm{Mor}(A, C)\ ,\ (f, g) \mapsto f \circ g,$$

such that the following axioms are fulfilled:

(1) If two pairs of objects (A, B) and (A', B') are not equal, then the sets $\mathrm{Mor}(A, B)$ and $\mathrm{Mor}(A', B')$ are disjoint.

(2) For every $A \in \mathrm{Obj}(\mathsf{A})$ there exists an element $\mathrm{id}_A \in \mathrm{Mor}(A, A)$ (called the *identity morphism* of A) such that for all $B \in \mathrm{Obj}(\mathsf{A})$, for all $f \in \mathrm{Mor}(B, A)$ and all $g \in \mathrm{Mor}(A, B)$ one has

$$\mathrm{id}_A \circ f = f \quad \text{and} \quad g \circ \mathrm{id}_A = g.$$

(3) The law of composition is *associative*, i.e., for any $A, B, C, D \in \mathrm{Obj}(\mathsf{A})$ and for any $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, $h \in \mathrm{Mor}(C, D)$ we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Examples A.1.2. a) In the category of sets Set the class of objects $\mathrm{Obj}(\mathsf{Set})$ consists of all sets, and for any two sets $A, B \in \mathrm{Obj}(\mathsf{Set})$ the set $\mathrm{Mor}(A, B)$ consists of all maps from A to B. Composition is the usual composition of maps.

b) The objects of the category Top are the topological spaces, and the morphisms are the continuous maps.

c) In the category of groups Groups one considers the class $\mathrm{Obj}(\mathsf{Groups})$ of all groups, and the morphisms are the group homomorphisms.

d) In AbelGr, the category of abelian groups, $\mathrm{Obj}(\mathsf{AbelGr})$ is the class of all abelian groups, and again the morphisms are the group homomorphisms.

Definition A.1.3. Let A and B be two categories. Then A is called a *full subcategory* of B provided

(1) $\mathrm{Obj}(\mathsf{A}) \subset \mathrm{Obj}(\mathsf{B})$,
(2) for any $A, B \in \mathrm{Obj}(\mathsf{A})$ the set of morphisms of A to B are the same in both categories A and B,
(3) for all $A, B, C \in \mathrm{Obj}(\mathsf{A})$, any $f \in \mathrm{Mor}(A, B)$ and any $g \in \mathrm{Mor}(B, C)$ the composites $g \circ f$ coincide in A and B,
(4) for $A \in \mathrm{Obj}(\mathsf{A})$ the identity morphism id_A is the same in both A and B.

Examples A.1.4. a) Top is not a full subcategory of Set because there are non-continuous maps between topological spaces.

b) AbelGr is a full subcategory of Groups.

Definition A.1.5. Let A and B be categories. A (*covariant*) *functor* T from A to B consists of a map $T\colon \mathrm{Obj}(\mathsf{A}) \to \mathrm{Obj}(\mathsf{B})$ and maps $T\colon \mathrm{Mor}(A, B) \to \mathrm{Mor}(TA, TB)$ for every $A, B \in \mathrm{Obj}(\mathsf{A})$ such that

(1) the composition is preserved, i.e., for all $A, B, C \in \mathrm{Obj}(\mathsf{A})$, for any $f \in \mathrm{Mor}(A, B)$ and for any $g \in \mathrm{Mor}(B, C)$ one has

$$T(g \circ f) = T(g) \circ T(f),$$

(2) T maps identities to identities, i.e., for any $A \in \mathrm{Obj}(\mathsf{A})$ we get

$$T(\mathrm{id}_A) = \mathrm{id}_{TA}.$$

In symbols one writes $T\colon \mathsf{A} \to \mathsf{B}$.

Examples A.1.6. a) For every category A one has the *identity functor* $\mathrm{Id}\colon \mathsf{A} \to \mathsf{A}$ which is defined by $\mathrm{Id}(A) = A$ for all $A \in \mathrm{Obj}(\mathsf{A})$ and $\mathrm{Id}(f) = f$ for all $f \in \mathrm{Mor}(A, B)$ with $A, B \in \mathrm{Obj}(\mathsf{A})$.

b) There is a functor $F\colon \mathsf{Top} \to \mathsf{Set}$ which maps each topological space to the underlying set and $F(g) = g$ for all $A, B \in \mathrm{Obj}(\mathsf{Top})$ and all $g \in \mathrm{Mor}(A, B)$. This functor F is called the *forgetful functor* because it forgets the topological structure.

c) Let A be a category. We fix an object $C \in \mathrm{Obj}(\mathsf{A})$. We define $T\colon \mathsf{A} \to \mathsf{Set}$ by $T(A) = \mathrm{Mor}(C, A)$ for all $A \in \mathrm{Obj}(\mathsf{A})$ and by

$$\begin{aligned} \mathrm{Mor}(A, B) &\to \mathrm{Mor}\big(\mathrm{Mor}(C, A), \mathrm{Mor}(C, B)\big), \\ f &\mapsto \big(g \mapsto f \circ g\big), \end{aligned}$$

for all $A, B \in \mathrm{Obj}(\mathsf{A})$. It is easy to check that T is a functor.

A.2 Functional analysis

In this section we give some background in functional analysis. More comprehensive expositions can be found e.g. in [Reed–Simon1980], [Reed–Simon1975], and [Rudin1973].

Definition A.2.1. A *Banach space* is a real or complex vector space X equipped with a norm $\|\cdot\|$ such that every Cauchy sequence in X has a limit.

Examples A.2.2. a) Consider $X = C^0([0,1])$, the space of continuous functions on the unit interval $[0,1]$. We pick the *supremum norm*: For $f \in C^0([0,1])$ one puts

$$\|f\|_{C^0([0,1])} := \sup_{t\in[0,1]} |f(t)|.$$

With this norm X is Banach space. In this example the unit interval can be replaced by any compact topological space.

b) More generally, let $k \in \mathbb{N}$ and let $X = C^k([0,1])$, the space of k times continuously differentiable functions on the unit interval $[0,1]$. The C^k*-norm* is defined by

$$\|f\|_{C^k([0,1])} := \max_{\ell=0,\dots,k} \|f^{(\ell)}\|_{C^0([0,1])}$$

where $f^{(\ell)}$ denotes the ℓ th derivative of $f \in X$. Then $X = C^k([0,1])$ together with the C^k-norm is a Banach space.

Now let H be a complex vector space, and let $(\cdot,\cdot)$ be a (positive definite) Hermitian scalar product. The scalar product induces a norm on H,

$$\|x\| := \sqrt{(x,x)} \quad \text{for all } x \in H.$$

Definition A.2.3. A complex vector space H endowed with Hermitian scalar product $(\cdot,\cdot)$ is called a *Hilbert space* if H together with the norm induced by $(\cdot,\cdot)$ forms a Banach space.

Example A.2.4. Consider the space of *square integrable functions* on $[0,1]$:

$$\mathcal{L}^2([0,1]) := \left\{ f : [0,1] \to \mathbb{C} \;\middle|\; f \text{ measurable and } \int_0^1 |f(t)|^2\, dt < \infty \right\}.$$

On $\mathcal{L}^2([0,1])$ one gets a natural sesquilinear form $(\cdot,\cdot)$ by $(f,g) := \int_0^1 \overline{f(t)} \cdot g(t)\, dt$ for all $f,g \in \mathcal{L}^2([0,1])$. Then $\mathcal{N} := \{f \in \mathcal{L}^2([0,1]) \mid (f,f) = 0\}$ is a linear subspace, and one denotes the quotient vector space by

$$L^2([0,1]) := \mathcal{L}^2([0,1]) \,/\, \mathcal{N}.$$

The sesquilinear form $(\cdot,\cdot)$ induces a Hermitian scalar product on $L^2([0,1])$. The Riesz–Fisher theorem [Reed–Simon1980, Example 2, p. 29] states that $L^2([0,1])$ equipped with this Hermitian scalar product is a Hilbert space.

Definition A.2.5. A *semi-norm* on a $\mathbb{K}$-vector space X, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is a map $\rho\colon X \to [0,\infty)$ such that

(1) $\rho(x+y) \le \rho(x) + \rho(y)$ for any $x, y \in X$,
(2) $\rho(\alpha\, x) = |\alpha|\, \rho(x)$ for any $x \in X$ and $\alpha \in \mathbb{K}$.

A family of semi-norms $\{\rho_i\}_{i\in I}$ is said to *separate points* if

(3) $\rho_i(x) = 0$ for all $i \in I$ implies $x = 0$.

Given a countable family of seminorms $\{\rho_k\}_{k\in\mathbb{N}}$ separating points one defines a metric d on X by setting for $x, y \in X$:

$$d(x,y) := \sum_{k=0}^{\infty} \frac{1}{2^k} \cdot \max\big(1, \rho_k(x-y)\big). \tag{A.15}$$

Definition A.2.6. A *Fréchet space* is a $\mathbb{K}$-vector space X equipped with a countable family of semi-norms $\{\rho_k\}_{k\in\mathbb{N}}$ separating points such that the metric d given by (A.15) is complete. The *natural topology* of a Fréchet space is the one induced by this metric d.

Example A.2.7. Let $C^\infty([0,1])$ be the space of smooth functions on the interval $[0,1]$. A countable family of semi-norms is given by the C^k-norms as defined in Example A.2.2 b). In order to prove that this family of (semi-)norms turns $C^\infty([0,1])$ into a Fréchet space we will show that $C^\infty([0,1])$ equipped with the metric d given by (A.15) is complete.

Let $(g_n)_n$ be a Cauchy sequence in $C^\infty([0,1])$ with respect to the metric d. Then for any $k \ge 0$ the sequence $(g_n)_n$ is Cauchy with respect to the C^k-norm. Since $C^k([0,1])$ together with the C^k-norm is a Banach space there exists a unique $h_k \in C^k([0,1]$ such that $(g_n)_n$ converges to h_k in the C^k-norm. From the estimate $\|\cdot\|_{C^k([0,1])} \le \|\cdot\|_{C^\ell([0,1])}$ for $k \le \ell$ we conclude that h_k and h_ℓ coincide. Therefore, putting $h := h_0$ we obtain $h \in C^\infty([0,1])$ and $d(h, g_n) \to 0$ for $n \to \infty$. This shows the completeness of $C^\infty([0,1])$.

If one wants to show that linear maps between Fréchet spaces are homeomorphisms, the following theorem is very helpful.

Theorem A.2.8 (Open Mapping Theorem). *Let X and Y be Fréchet spaces, and let $f\colon X \to Y$ be a continuous linear surjection. Then f is open. In particular, continuous linear bijections between Fréchet spaces are homeomorphisms.*

Proof. See [Rudin1973, Cor. 2.12., p. 48] or [Reed–Simon1980, Thm. V.6, p. 132] □

From now on we fix a Hilbert space H. A continuous linear map $H \to H$ is called *bounded operator* on H. But many operators occurring in analysis and mathematical physics are not continuous and not even defined on the whole Hilbert space. Therefore one introduces the concept of unbounded operators.

Definition A.2.9. Let $\mathrm{dom}(A) \subset H$ be a linear subspace of H. A linear map $A\colon \mathrm{dom}(A) \to H$ is called an *unbounded operator* in H with *domain* $\mathrm{dom}(A)$. One says that A is *densely defined* if $\mathrm{dom}(A)$ is a dense subspace of H.

Example A.2.10. One can represent elements of $L^2(\mathbb{R})$ by functions. The space of smooth functions with compact support $C_c^\infty(\mathbb{R})$ is regarded as a linear subspace, $C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$. Then one can consider the differentiation operator $A := \frac{d}{dt}$ as an unbounded operator in $L^2(\mathbb{R})$ with domain $\mathrm{dom}(A) = C_c^\infty(\mathbb{R})$, and A is densely defined.

Definition A.2.11. Let A be an unbounded operator on H with domain $\mathrm{dom}(A)$. The *graph* of A is the set

$$\Gamma(A) := \{(x, Ax) \mid x \in \mathrm{dom}(A)\} \subset H \times H.$$

The operator A is called a *closed* operator if its graph $\Gamma(A)$ is a closed subset of $H \times H$.

Definition A.2.12. Let A_1 and A_2 be operators on H. If $\mathrm{dom}(A_1) \supset \mathrm{dom}(A_2)$ and $A_1 x = A_2 x$ for all $x \in \mathrm{dom}(A_2)$, then A_1 is said to be an *extension* of A_2. One then writes $A_1 \supset A_2$.

Definition A.2.13. Let A be an operator on H. An operator A is *closable* if it possesses a closed extension. In this case the closure $\overline{\Gamma(A)}$ of $\Gamma(A)$ in $H \times H$ is the graph of an operator called the *closure* of A.

Definition A.2.14. Let A be a densely defined operator on H. Then we put

$$\mathrm{dom}(A^*) := \{x \in H \mid \text{there is a } y \in H \text{ with } (Az, x) = (z, y) \text{ for all } z \in \mathrm{dom}(A)\}.$$

For each $x \in \mathrm{dom}(A^*)$ we define $A^* x := y$ where y is uniquely determined by the requirement $(Az, x) = (z, y)$ for all $z \in \mathrm{dom}(A)$. Uniqueness of y follows from $\mathrm{dom}(A)$ being dense in H. We call A^* the *adjoint* of A.

Definition A.2.15. A densely defined operator A on H is called *symmetric* if A^* is an extension of A, i.e., if $\mathrm{dom}(A) \subset \mathrm{dom}(A^*)$ and $Ax = A^* x$ for all $x \in \mathrm{dom}(A)$. The operator A is called *selfadjoint* if $A = A^*$, that is, if A is symmetric and $\mathrm{dom}(A) = \mathrm{dom}(A^*)$.

Any symmetric operator is closable with closure $\bar{A} = A^{**}$.

Definition A.2.16. A symmetric operator A is called *essentially selfadjoint* if its closure $\bar{A}$ is selfadjoint.

We conclude this section by stating a criterion for essential selfadjointness of a symmetric operator.

Definition A.2.17. Let A be an operator on a Hilbert space H. Then one calls the set $C^\infty(A) := \bigcap_{n=1}^\infty \mathrm{dom}(A^n)$ the set of C^∞-*vectors* for A. A vector $\varphi \in C^\infty(A)$ is called an *analytic vector* for A if

$$\sum_{n=0}^{\infty} \frac{\|A^n \varphi\|}{n!} t^n < \infty \quad \text{for some } t > 0.$$

Theorem A.2.18 (Nelson's Theorem). *Let A be a symmetric operator on a Hilbert space H. If* $\mathrm{dom}(A)$ *contains a set of analytic vectors which is dense in H, then A is essentially selfadjoint.*

Proof. See [Reed–Simon1975, Thm. X.39, p. 202]. □

If A is a selfadjoint operator and $f : \mathbb{R} \to \mathbb{C}$ is a bounded Borel-measurable function, then one can define the bounded operator $f(A)$ in a natural manner. We use this to get the unitary operator $\exp(iA)$ in Section 4.6. If φ is an analytic vector, then

$$\exp(iA)\varphi = \sum_{n=0}^{\infty} \frac{i^n}{n!} A^n \varphi.$$

A.3 Differential geometry

In this section we introduce the basic geometrical objects such as manifolds and vector bundles which are used throughout the text. A detailed introduction can be found e.g. in [Spivak1979] or in [Nicolaescu1996].

A.3.1 Differentiable manifolds. We start with the concept of a manifold. Loosely speaking, manifolds are spaces which locally look like $\mathbb{R}^n$.

Definition A.3.1. Let n be an integer. A topological space M is called an *n-dimensional topological manifold* if and only if

(1) its topology is Hausdorff and has a countable basis, and
(2) it is locally homeomorphic to $\mathbb{R}^n$, i.e., for every $p \in M$ there exists an open neighborhood U of p in M and a homeomorphism $\varphi : U \to \varphi(U)$, where $\varphi(U)$ is an open subset of $\mathbb{R}^n$.

Any such homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is called a *(local) chart* of M. The coordinate functions $\varphi^j : U \to \mathbb{R}$ of $\varphi = (\varphi^1, \ldots, \varphi^n)$ are called the *coordinates* of the local chart. An *atlas* of M is a family of local charts $(U_i, \varphi_i)_{i \in I}$ of M which covers all of M, i.e., $\bigcup_{i \in I} U_i = M$.

Definition A.3.2. Let M be a topological manifold. A *smooth atlas* of M is an atlas $(U_i, \varphi_i)_{i \in I}$ such that

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is a smooth map (as a map between open subsets of $\mathbb{R}^n$) whenever $U_i \cap U_j \neq \emptyset$.

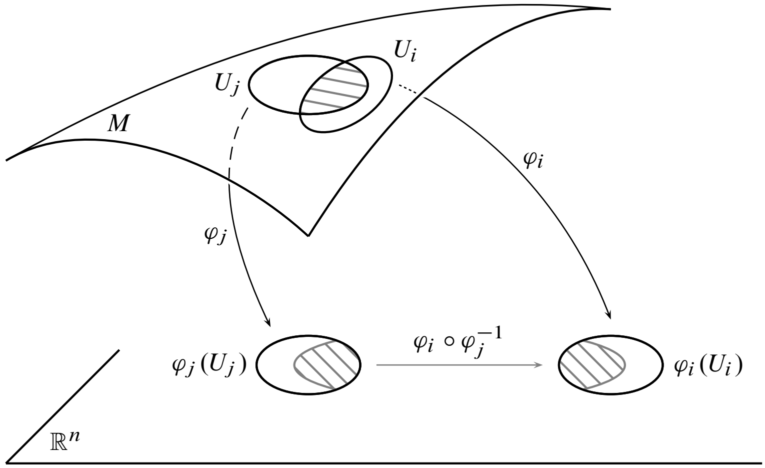


Figure 38. Smooth atlas.

Not every topological manifold admits a smooth atlas. We shall only be interested in those topological manifolds that do. Moreover, topological manifolds can have essentially different smooth atlases in the sense that they give rise to non-diffeomorphic smooth manifolds. Hence the smooth atlas is an important additional piece of structure.

Definition A.3.3. A *smooth manifold* is a topological manifold M together with a maximal smooth atlas.

Maximality means that there is no smooth atlas on M containing all local charts of the given atlas except for the given atlas itself. Every smooth atlas is contained in a unique maximal smooth atlas.

In the following "manifold" will always mean "smooth manifold". The smooth atlas will usually be suppressed in the notation.

Examples A.3.4. a) Every nonempty open subset of $\mathbb{R}^n$ is an n-dimensional manifold. More generally, any nonempty open subset of an n-dimensional manifold is itself an n-dimensional manifold.

b) The product of any m-dimensional manifold with any n-dimensional manifold is canonically an $(m+n)$-dimensional manifold.

c) Let $n \leq m$. An n-dimensional *submanifold* N of an m-dimensional manifold M is a nonempty subset N of M such that for every $p \in N$ there exists a local chart (U, φ) of M about p with

$$\varphi(U \cap N) = \varphi(U) \cap \mathbb{R}^n,$$

where we identify $\mathbb{R}^n \cong \mathbb{R}^n \times \{0\} \subset \mathbb{R}^m$. Any submanifold is canonically a manifold. In the case $n = m - 1$ the submanifold N is called *hypersurface* of M.

As in the case of open subsets of $\mathbb{R}^n$, we have the concept of *differentiable map* between manifolds:

Definition A.3.5. Let M and N be manifolds and let $p \in M$. A continuous map $f: M \to N$ is said to be *differentiable* at the point p if there exist local charts (U, φ) and (V, ψ) about p in M and about $f(p)$ in N respectively, such that $f(U) \subset V$ and

$$\psi \circ f \circ \varphi^{-1}: \varphi(U) \to \psi(V)$$

is differentiable at $\varphi(p) \in \varphi(U)$. The map f is said to be differentiable on M if it is differentiable at every point of M.

Similarly, one defines C^k-maps between smooth manifolds, $k \in \mathbb{N} \cup \{\infty\}$. A C^∞-map is also called a *smooth map*.

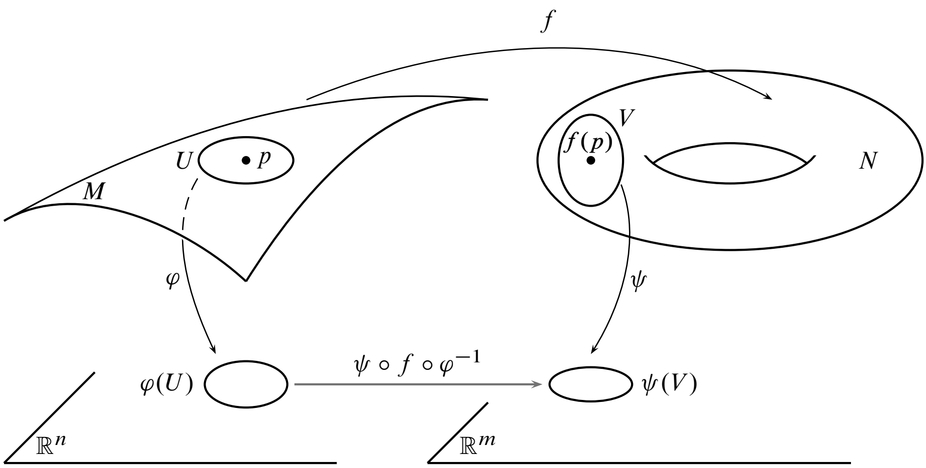


Figure 39. Differentiability of a map.

Note that, if $\psi \circ f \circ \varphi^{-1}$ is (C^k-)differentiable for *some* local charts φ, ψ as in Definition A.3.5, then so is $\psi' \circ f \circ \varphi'^{-1}$ for *any* other pair of local charts φ', ψ' obeying the same conditions. This is a consequence of the fact that the atlases of M and N have been assumed to be smooth.

In order to define the differential of a differentiable map between manifolds, we need the concept of *tangent space*:

Definition A.3.6. Let M be a manifold and $p \in M$. Consider the set $\mathcal{T}_p$ of differentiable curves $c: I \to M$ with $c(0) = p$ where I is an open interval containing $0 \in \mathbb{R}$. The *tangent space* of M at p is the quotient

$$T_pM := \mathcal{T}_p / \sim,$$

where "$\sim$" is the equivalence relation defined as follows: Two smooth curves $c_1, c_2 \in \mathcal{T}_p$ are equivalent if and only if there exists a local chart about p such that $(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$.

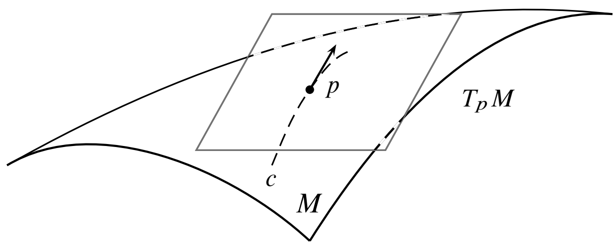


Figure 40. Tangent space T_pM.

One checks that the definition of the equivalence relation does not depend on the choice of local chart: If $(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$ for one local chart (U, φ) with $p \in U$, then $(\psi \circ c_1)'(0) = (\psi \circ c_2)'(0)$ for all local charts (V, ψ) with $p \in V$.

Let n denote the dimension of M. Denote the equivalence class of $c \in \mathcal{T}_p$ in T_pM by $[c]$. It can be easily shown that the map

$$\begin{aligned}\Theta_\varphi \colon T_pM &\to \mathbb{R}^n,\\ [c] &\mapsto (\varphi \circ c)'(0),\end{aligned}$$

is a well-defined bijection. Hence we can introduce a vector space structure on T_pM by *declaring* Θ_φ to be a linear isomorphism. This vector space structure is independent of the choice of local chart because for two local charts (U, φ) and (V, ψ) containing p the map $\Theta_\psi \circ \Theta_\varphi^{-1} = d_{\varphi(p)}(\psi \circ \varphi^{-1})$ is linear.

By definition, the *tangent bundle* of M is the disjoint union of all the tangent spaces of M,

$$TM := \dot{\bigcup_{p \in M}} T_pM.$$

Definition A.3.7. Let $f : M \to N$ be a differentiable map between manifolds and let $p \in M$. The *differential* of f at p (also called the *tangent map* of f at p) is the map

$$d_pf : T_pM \to T_{f(p)}N, \quad [c] \mapsto [f \circ c].$$

The *differential* of f is the map $df : TM \to TN$, $df_{|T_pM} := d_pf$.

The map d_pf is well defined and linear. The map f is said to be an *immersion* or a *submersion* if d_pf is injective or surjective for every $p \in M$ respectively. A *diffeomorphism* between manifolds is a smooth bijective map whose inverse is also smooth. An *embedding* is an immersion $f : M \to N$ such that $f(M) \subset N$ is a submanifold of N and $f : M \to f(M)$ is a diffeomorphism.

Using local charts basically all local properties of differential calculus on $\mathbb{R}^n$ can be translated to manifolds. For example, we have the *chain rule*

$$d_p(g \circ f) = d_{f(p)}g \circ d_pf,$$

and the *inverse function theorem* which states that if $d_pf : T_pM \to T_{f(p)}N$ is a linear isomorphism, then f maps a neighborhood of p diffeomorphically onto a neighborhood of $f(p)$.

A.3.2 Vector bundles. We can think of the tangent bundle as a family of pairwise disjoint vector spaces parametrized by the points of the manifold. In a suitable sense these vector spaces depend smoothly on the base point. This is formalized by the concept of a vector bundle.

Definition A.3.8. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let E and M be manifolds of dimension $m+n$ and m respectively. Let $\pi\colon E \to M$ be a surjective smooth map. Let the fiber $E_p := \pi^{-1}(p)$ carry a structure of $\mathbb{K}$-vector space $\mathcal{V}_p$ for each $p \in M$. The quadruple $(E, \pi, M, \{\mathcal{V}_p\}_{p\in M})$ is called a $\mathbb{K}$-*vector bundle* if for every $p \in M$ there exists an open neighborhood U of p in M and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{K}^n$ such that the following diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \Phi\ } & U \times \mathbb{K}^n \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & U & \end{array} \tag{A.16}$$

commutes and for every $q \in U$ the map $\pi_2 \circ \Phi|_{E_q}\colon E_q \to \mathbb{K}^n$ is a vector space isomorphism. Here $\pi_1\colon U \times \mathbb{K}^n \to U$ denotes the projection onto the first factor U and $\pi_2\colon U \times \mathbb{K}^n \to \mathbb{K}^n$ is the projection onto the second factor $\mathbb{K}^n$.

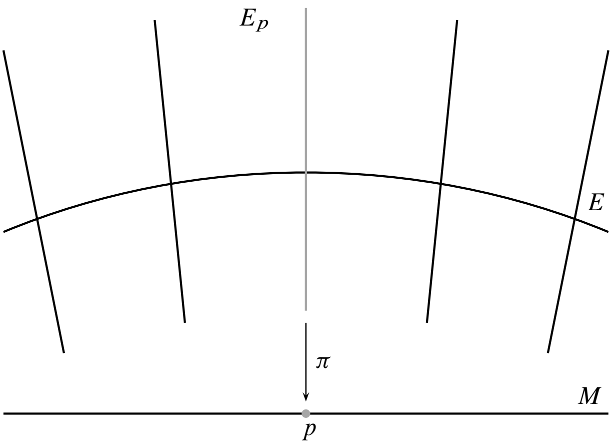


Figure 41. Vector bundle.

Such a map $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{K}^n$ is called a *local trivialization* of the vector bundle. The manifold E is called the *total space*, M the *base*, and the number n the *rank* of the vector bundle. Often one simply speaks of the vector bundle E for brevity.

A vector bundle is said to be *trivial* if it admits a global trivialization, that is, if there exists a diffeomorphism as in (A.16) with $U = M$.

Examples A.3.9. a) The tangent bundle of any n-dimensional manifold M is a real vector bundle of rank n. The map π is given by the canonical map $\pi(T_p M) = \{p\}$ for all $p \in M$.

b) Most operations from linear algebra on vector spaces can be carried out fiberwise on vector bundles to give new vector bundles. For example, for a given vector bundle E one can define the *dual vector bundle* E^*. Here one has by definition $(E^*)_p = (E_p)^*$. Similarly, one can define the exterior and the symmetric powers of E. For given $\mathbb{K}$-vector bundles E and F one can form the direct sum $E \oplus F$, the tensor product $E \otimes F$, the bundle $\mathrm{Hom}_{\mathbb{K}}(E, F)$ etc.

c) The dual vector bundle of the tangent bundle is called the *cotangent bundle* and is denoted by T^*M.

d) Let n be the dimension of M and $k \in \{0, 1, \dots, n\}$. The kth exterior power of T^*M is the *bundle of k-linear skew-symmetric forms* on TM and is denoted by $\Lambda^k T^*M$. It is a real vector bundle of rank $\frac{n!}{k!(n-k)!}$. By convention $\Lambda^0 T^*M$ is the trivial real vector bundle of rank 1.

Definition A.3.10. A *section* in a vector bundle $(E, \pi, M, \{\mathcal{V}_p\}_{p \in M})$ is a map $s\colon M \to E$ such that

$$\pi \circ s = \mathrm{id}_M.$$

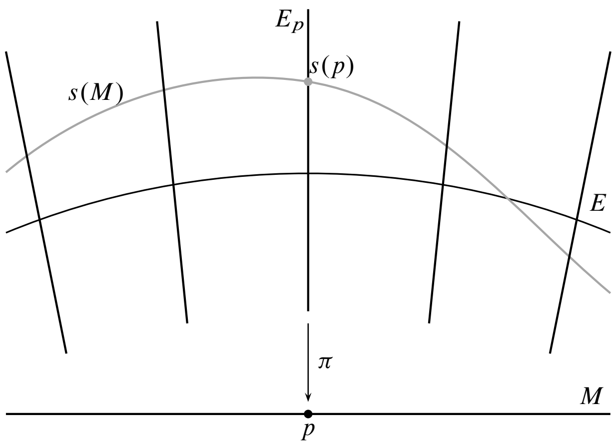


Figure 42. Section in a vector bundle.

As M and E are smooth manifolds we can speak about C^k-sections, $k \in \mathbb{N} \cup \{\infty\}$. The set $C^k(M, E)$ of C^k-sections of a given $\mathbb{K}$-vector bundle forms a $\mathbb{K}$-vector space, and a module over the algebra $C^k(M, \mathbb{K})$ as well because multiplying *pointwise* any C^k-section with any C^k-function one obtains a new C^k-section.

In each vector bundle there exists a canonical smooth section, namely the *zero section* defined by $s(x) := 0_x \in E_x$. However, there does not in general exist any smooth *nowhere vanishing* section. Moreover, the existence of n everywhere linearly independent smooth sections in a vector bundle of rank n is equivalent to its *triviality*.

Examples A.3.11. a) Let $E = M \times \mathbb{K}^n$ be the trivial $\mathbb{K}$-vector bundle of rank n over M. Then the sections of E are essentially just the $\mathbb{K}^n$-valued functions on M.

b) The sections in $E = TM$ are called the *vector fields* on M. If (U, φ) is a local chart of the n-dimensional manifold M, then for each $j = 1, \dots, n$ the curve $c(t) = \varphi^{-1}(\varphi(p) + te_j)$ represents a tangent vector $\frac{\partial}{\partial x^j}(p)$ where $e_1, \dots, e_n$ denote the standard basis of $\mathbb{R}^n$. The vector fields $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ are smooth on U and yield a basis of T_pM for every $p \in U$,

$$T_pM = \mathrm{Span}_{\mathbb{R}}\left(\frac{\partial}{\partial x^1}(p), \dots, \frac{\partial}{\partial x^n}(p)\right).$$

c) The sections in $E = T^*M$ are called the 1*-forms*. Let (U, φ) be a local chart of M. Denote the basis of T_p^*M dual to $\frac{\partial}{\partial x^1}(p), \dots, \frac{\partial}{\partial x^n}(p)$ by $dx^1(p), \dots, dx^n(p)$. Then $dx^1, \dots, dx^n$ are smooth 1-forms on U.

d) Fix $k \in \{0, \dots, n\}$. Sections in $E = \Lambda^k T^*M$ are called k*-forms*. Given a local chart (U, φ) we get smooth k-forms which pointwise yield a basis of $\Lambda^k T^*M$ by

$$dx^{i_1} \wedge \dots \wedge dx^{i_k}, \quad 1 \le i_1 < \dots < i_k \le n.$$

In particular, for $k = n$ the bundle $\Lambda^n T^*M$ has rank 1 and a local chart yields the smooth local section $dx^1 \wedge \dots \wedge dx^n$. Existence of a global smooth section in $\Lambda^n T^*M$ is equivalent to M being orientable.

e) For each $p \in M$ let $|\Lambda M|_p$ be the set of all functions $v\colon \Lambda^n T_p^*M \to \mathbb{R}$ with $v(\lambda X) = |\lambda| \cdot v(X)$ for all $X \in \Lambda^n T_p^*M$ and all $\lambda \in \mathbb{R}$. Now $|\Lambda M|_p$ is a 1-dimensional real vector space and yields a vector bundle $|\Lambda M|$ of rank 1 over M. Sections in $|\Lambda M|$ are called *densities*.

Given a local chart (U, φ) there is a smooth density $|dx|$ defined on U and characterized by

$$|dx|(dx^1 \wedge \dots \wedge dx^n) = 1.$$

The bundle $|\Lambda M|$ is always trivial. Its importance lies in the fact that densities can be integrated. There is a unique linear map

$$\int_M : \mathcal{D}(M, |\Lambda M|) \to \mathbb{R},$$

called the *integral*, such that for any local chart (U, φ) and any $f \in \mathcal{D}(U, \mathbb{R})$ we have

$$\int_M f\,|dx| = \int_{\varphi(U)} (f \circ \varphi^{-1})(x^1, \dots, x^n)\, dx^1 \dots dx^n$$

where the right-hand side is the usual integral of functions on $\mathbb{R}^n$ and $\mathcal{D}(M, E)$ denotes the set of smooth sections with compact support.

f) Let E be a real vector bundle. Smooth sections in $E^* \otimes E^*$ which are pointwise *nondegenerate symmetric* bilinear forms are called *semi-Riemannian metrics* or *inner products* on E. An inner product on E is called *Riemannian metric* if it is pointwise positive definite. An inner product on E is called a *Lorentzian metric* if it has pointwise signature $(- + \ldots +)$. In case $E = TM$ a Riemannian or Lorentzian metric on E is also called a Riemannian or Lorentzian metric on M respectively. A *Riemannian* or *Lorentzian manifold* is a manifold M together with a Riemannian or Lorentzian metric on M respectively.

Any semi-Riemannian metric on T^*M yields a nowhere vanishing smooth density dV on M. In local coordinates, write the semi-Riemannian metric as

$$\sum_{i,j=1}^{n} g_{ij}\, dx^i \otimes dx^j .$$

Then the induced density is given by

$$\mathrm{dV} = \sqrt{|\det(g_{ij})|}\, |dx|.$$

Therefore there is a canonical way to form the integral $\int_M f\, \mathrm{dV}$ of any function $f \in \mathcal{D}(M)$ on a Riemannian or Lorentzian manifold.

g) If E is a complex vector bundle a *Hermitian metric* on E is by definition a smooth section of $E^* \otimes_{\mathbb{R}} E^*$ (the *real* tensor product of E^* with itself) which is a Hermitian scalar product on each fiber.

Definition A.3.12. Let $(E, \pi, M, \{\mathcal{V}_p\}_{p\in M})$ and $(E', \pi', M', \{\mathcal{V}'_{p'}\}_{p'\in M'})$ be $\mathbb{K}$-vector bundles. A *vector-bundle-homomorphism* from E to E' is a pair (f, F) where

(1) $f : M \to M'$ is a smooth map,

(2) $F : E \to E'$ is a smooth map such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

commutes and such that $F_{|E_p} : E_p \to E'_{f(p)}$ is $\mathbb{K}$-linear for every $p \in M$.

If $M = M'$ and $f = \mathrm{id}_M$ a vector-bundle-homomorphism is simply a smooth section in $\mathrm{Hom}_{\mathbb{K}}(E, E') \to M$.

Remark A.3.13. Let E be a real vector bundle with inner product $\langle \cdot, \cdot \rangle$. Then we get a vector-bundle-isomorphism ☞: $E \to E^*$, ☞$(X) = \langle X, \cdot \rangle$.

In particular, on a Riemannian or Lorentzian manifold M with $E = TM$ one can define the *gradient* of a differentiable function $f : M \to \mathbb{R}$ by grad $f :=$ ☞$^{-1}(df) = (df)^\sharp$. The differential df is a 1-form defined independently of the metric while the gradient grad f is a vector field whose definition does depend on the semi-Riemannian metric.

A.3.3 Connections on vector bundles. For a differentiable function $f: M \to \mathbb{R}$ on a (smooth) manifold M its derivative in direction $X \in C^\infty(M, TM)$ is defined by

$$\partial_X f := df(X).$$

We have defined the concept of differentiability of a section s in a vector bundle. What is the derivative of s?

Without further structure there is no canonical way of defining this. A rule for differentiation of sections in a vector bundle is called a connection.

Definition A.3.14. Let $(E, \pi, M, \{\mathcal{V}_p\}_{p \in M})$ be a $\mathbb{K}$-vector bundle, $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. A *connection* (or *covariant derivative*) on E is a $\mathbb{R}$-bilinear map

$$\begin{aligned} \nabla: C^\infty(M, TM) \times C^\infty(M, E) &\to C^\infty(M, E), \\ (X, s) &\mapsto \nabla_X s, \end{aligned}$$

with the following properties:

(1) The map ∇ is $C^\infty(M)$-*linear* in the first argument, i.e.,

$$\nabla_{fX} s = f \nabla_X s$$

holds for all $f \in C^\infty(M)$, $X \in C^\infty(M, TM)$ and $s \in C^\infty(M, E)$.

(2) The map ∇ is a *derivation* with respect to the second argument, i.e., it is $\mathbb{K}$-bilinear and

$$\nabla_X (f \cdot s) = \partial_X f \cdot s + f \cdot \nabla_X s$$

holds for all $f \in C^\infty(M)$, $X \in C^\infty(M, TM)$ and $s \in C^\infty(M, E)$.

The properties of a connection imply that the value of $\nabla_X s$ at a given point $p \in M$ depends only on $X(p)$ and on the values of s on a curve representing $X(p)$.

Let ∇ be a connection on a vector bundle E over M. Let $c: [a, b] \to M$ be a smooth curve. Given $s_0 \in E_{c(a)}$ there is a unique smooth solution $s: [a, b] \to E$, $t \mapsto s(t) \in E_{c(t)}$, satisfying $s(a) = s_0$ and

$$\nabla_{\dot{c}} s = 0. \tag{A.17}$$

This follows from the fact that in local coordinates (A.17) is a linear ordinary differential equation of first order. The map

$$\Pi_c: E_{c(a)} \to E_{c(b)}, \quad s_0 \mapsto s(b),$$

is called *parallel transport*. It is easy to see that Π_c is a linear isomorphism. This shows that a connection allows us via its parallel transport to “connect” different fibers of the vector bundle. This is the origin of the term “connection”. Be aware that in general the parallel transport Π_c does depend on the choice of curve c connecting its endpoints.

Any connection ∇ on a vector bundle E induces a connection, also denoted by ∇, on the dual vector bundle E^* by

$$(\nabla_X \theta)(s) := \partial_X (\theta(s)) - \theta (\nabla_X s)$$

for all $X \in C^\infty(M, TM)$, $\theta \in C^\infty(M, E^*)$ and $s \in C^\infty(M, E)$. Here $\theta(s) \in C^\infty(M)$ is the function on M obtained by pointwise evaluation of $\theta(p) \in E_p^*$ on $s(p) \in E_p$.

Similarly, tensor products, exterior and symmetric products, and direct sums inherit connections from the connections on the vector bundles out of which they are built. For example, two connections ∇ and ∇' on E and E' respectively induce a connection D on $E \otimes E'$ by

$$D_X(s \otimes s') := (\nabla_X s) \otimes s' + s \otimes (\nabla'_X s')$$

and a connection $\tilde{D}$ on $E \oplus E'$ by

$$\tilde{D}_X(s \oplus s') := (\nabla_X s) \oplus (\nabla'_X s')$$

for all $X \in C^\infty(M, TM)$, $s \in C^\infty(M, E)$ and $s' \in C^\infty(M, E')$.

If a vector bundle E carries a semi-Riemannian or Hermitian metric $\langle \cdot\,, \cdot \rangle$, then a connection ∇ on E is called *metric* if the following Leibniz rule holds:

$$\partial_X(\langle s, s' \rangle) = \langle \nabla_X s, s' \rangle + \langle s, \nabla_X s' \rangle$$

for all $X \in C^\infty(M, TM)$ and $s, s' \in C^\infty(M, E)$.

Given two vector fields $X, Y \in C^\infty(M, TM)$ there is a unique vector field $[X, Y] \in C^\infty(M, TM)$ characterized by

$$\partial_{[X,Y]} f = \partial_X \partial_Y f - \partial_Y \partial_X f$$

for all $f \in C^\infty(M)$. The map $[\cdot\,, \cdot]\colon C^\infty(M, TM) \times C^\infty(M, TM) \to C^\infty(M, TM)$ is called the *Lie bracket*. It is $\mathbb{R}$-bilinear, skew-symmetric and satisfies the *Jacobi identity*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Definition A.3.15. Let ∇ be a connection on a vector bundle E. The *curvature tensor* of ∇ is the map

$$R\colon C^\infty(M, TM) \times C^\infty(M, TM) \times C^\infty(M, E) \to C^\infty(M, E),$$

$$(X, Y, s) \mapsto R(X, Y)s := \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s.$$

One can check that the value of $R(X, Y)s$ at any point $p \in M$ depends only on $X(p)$, $Y(p)$, and $s(p)$. Thus the curvature tensor can be regarded as a section, $R \in C^\infty(M, \Lambda^2 T^*M \otimes \mathrm{Hom}_{\mathbb{K}}(E, E))$.

Now let M be an n-dimensional manifold with semi-Riemannian metric g on TM. It can be shown that there exists a unique metric connection ∇ on TM satisfying

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all vector fields X and Y on M. This connection is called the *Levi-Civita* connection of the semi-Riemannian manifold (M, g). Its curvature tensor R is the *Riemannian curvature tensor* of (M, g). The *Ricci curvature* $\mathrm{ric} \in C^\infty(M, T^*M \otimes T^*M)$ is defined by

$$\mathrm{ric}(X, Y) := \sum_{j=1}^{n} \varepsilon_j \, g(R(X, e_j)e_j, Y)$$

where $e_1, \dots, e_n$ are smooth locally defined vector fields which are pointwise orthonormal with respect to g and $\varepsilon_j = g(e_j, e_j) = \pm 1$. It can easily be checked that this definition is independent of the choice of the vector fields $e_1, \dots, e_n$. Similarly, the *scalar curvature* is the function $\mathrm{scal} \in C^\infty(M, \mathbb{R})$ defined by

$$\mathrm{scal} := \sum_{j=1}^{n} \varepsilon_j \ \mathrm{ric}(e_j, e_j).$$

A.4 Differential operators

In this section we explain the concept of linear differential operators and we define the principal symbol. A detailed introduction to the topic can be found e.g. in [Nicolaescu1996, Ch. 9]. As before we write $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition A.4.1. Let E and F be $\mathbb{K}$-vector bundles of rank n and m respectively over a d-dimensional manifold M. A *linear differential operator of order at most k* from E to F is a $\mathbb{K}$-linear map

$$L \colon C^\infty(M, E) \to C^\infty(M, F)$$

which can locally be described as follows: For every $p \in M$ there exists an open coordinate-neighborhood U of p in M on which E and F are trivialized and there are smooth maps $A_\alpha \colon U \to \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^m)$ such that on U

$$Ls = \sum_{|\alpha| \le k} A_\alpha \frac{\partial^{|\alpha|} s}{\partial x^\alpha}.$$

Here summation is taken over all multiindices $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ with $|\alpha| := \sum_{r=1}^{d} \alpha_r \le k$. Moreover, $\frac{\partial^{|\alpha|}}{\partial x^\alpha} := \frac{\partial^{\alpha_1 + \dots + \alpha_d}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$. In this definition we have used the local trivializations to identify sections in E with $\mathbb{K}^n$-valued functions and sections in F with $\mathbb{K}^m$-valued functions on U. If L is a linear differential operator of order at most k, but not of order at most $k - 1$, then we say that L is of *order* k.

Note that zero-order differential operators are nothing but sections of $\mathrm{Hom}_{\mathbb{K}}(E, F)$, i.e., they are vector-bundle-homomorphisms from E to F.

Definition A.4.2. Let L be a linear differential operator of order k from E to F. The *principal symbol* of L is the map

$$\sigma_L : T^*M \to \mathrm{Hom}_{\mathbb{K}}(E, F)$$

defined locally as follows: For a given $p \in M$ write $L = \sum_{|\alpha| \le k} A_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ in a coordinate neighborhood of p with respect to local trivializations of E and F as in Definition A.4.1. For every $\xi = \sum_{r=1}^{d} \xi_r \cdot dx^r \in T_p^*M$ we have with respect to these trivializations,

$$\sigma_L(\xi) := \sum_{|\alpha|=k} \xi^\alpha A_\alpha(p)$$

where $\xi^\alpha := \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d}$. Here we have used the local trivializations of E and F to identify $\mathrm{Hom}_{\mathbb{K}}(E, F)$ with $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^m)$.

One can show that the principal symbol of a differential operator is well defined, that is, it is independent of the choice of the local coordinates and trivializations. Moreover, the principal symbol of a differential operator of order k is, by definition, a homogeneous polynomial of degree k on T^*M.

Example A.4.3. The gradient is a linear differential operator of first order

$$\mathrm{grad} : C^\infty(M, \mathbb{R}) \to C^\infty(M, TM)$$

with principal symbol

$$\sigma_{\mathrm{grad}}(\xi) f = f \cdot \xi^\sharp.$$

Example A.4.4. The divergence yields a first order linear differential operator

$$\mathrm{div} : C^\infty(M, TM) \to C^\infty(M, \mathbb{R})$$

with principal symbol

$$\sigma_{\mathrm{div}}(\xi) X = \xi(X).$$

Example A.4.5. For each $k \in \mathbb{N}$ there is a unique linear first order differential operator

$$d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M),$$

called *exterior differential*, such that

(1) for $k = 0$ the exterior differential coincides with the differential defined in Definition A.3.7, after the canonical identification $T_y\mathbb{R} = \mathbb{R}$,
(2) $d^2 = 0 : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+2} T^*M)$ for all k,
(3) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$ for all $\omega \in C^\infty(M, \Lambda^k T^*M)$ and $\eta \in C^\infty(M, \Lambda^l T^*M)$.

Its principal symbol is given by

$$\sigma_d(\xi)\,\omega = \xi \wedge \omega.$$

Example A.4.6. A connection ∇ on a vector bundle E can be considered as a first order linear differential operator

$$\nabla\colon C^\infty(M,E) \to C^\infty(M,T^*M \otimes E).$$

Its principal symbol is easily be seen to be

$$\sigma_\nabla(\xi)\,e = \xi \otimes e.$$

Example A.4.7. If L is a linear differential operator of order 0, i.e., L is an element of $C^\infty(M,\mathrm{Hom}(E,F))$, then

$$\sigma_L(\xi) = L.$$

Remark A.4.8. If $L_1\colon C^\infty(M,E) \to C^\infty(M,F)$ is a linear differential operator of order k and $L_2\colon C^\infty(M,F) \to C^\infty(M,G)$ is a linear differential operator of order l, then $L_2 \circ L_1$ is a linear differential operator of order $k+l$. The principal symbols satisfy

$$\sigma_{L_2\circ L_1}(\xi) = \sigma_{L_2}(\xi) \circ \sigma_{L_1}(\xi).$$

A.5 More on Lorentzian geometry

This section is a rather heterogeneous collection of results on Lorentzian manifolds. We give full proofs. This material has been collected in this appendix in order not to overload Section 1.3 with technical statements.

Throughout this section M denotes a Lorentzian manifold.

Lemma A.5.1. *Let the causal relation $\leq$ on M be closed, i.e., for all convergent sequences $p_n \to p$ and $q_n \to q$ in M with $p_n \leq q_n$ we have $p \leq q$.*

Then for every compact subset K of M the subsets $J_+^M(K)$ and $J_-^M(K)$ are closed.

Proof. Let $(q_n)_{n\in\mathbb{N}}$ be any sequence in $J_+^M(K)$ converging in M and $q \in M$ be its limit. By definition, there exists a sequence $(p_n)_{n\in\mathbb{N}}$ in K with $p_n \leq q_n$ for every n. Since K is compact we may assume, after to passing to a subsequence, that $(p_n)_{n\in\mathbb{N}}$ converges to some $p \in K$. Since $\leq$ is closed we get $p \leq q$, hence $q \in J_+^M(K)$. This shows that $J_+^M(K)$ is closed. The proof for $J_-^M(K)$ is the same. □

Remark A.5.2. If K is only assumed to be *closed* in Lemma A.5.1, then $J_\pm^M(K)$ need not be closed. The following picture shows a curve K, closed as a subset and asymptotic to a lightlike line in 2-dimensional Minkowski space. Its causal future $J_+^M(K)$ is the open half plane bounded by this lightlike line.

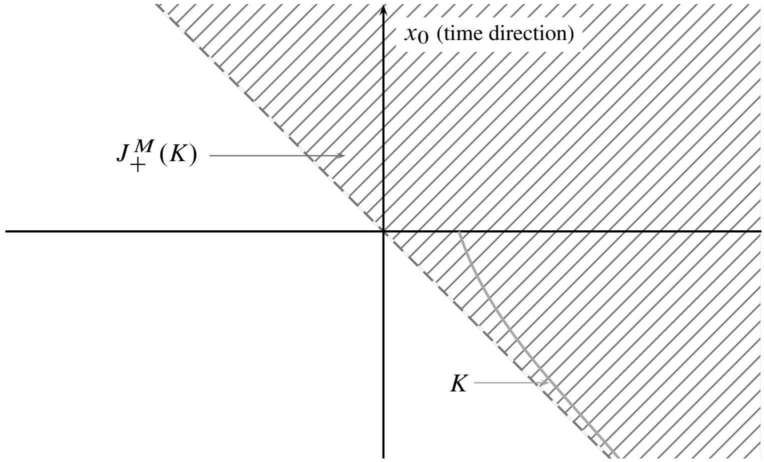


Figure 43. Causal future $J_+^M(K)$ is open.

Lemma A.5.3. *Let M be a time-oriented Lorentzian manifold. Let $K \subset M$ be a compact subset. Let $A \subset M$ be a subset such that, for every $x \in M$, the intersection $A \cap J_-^M(x)$ is relatively compact in M.*

Then $A \cap J_-^M(K)$ is a relatively compact subset of M. Similarly, if $A \cap J_+^M(x)$ is relatively compact for every $x \in M$, then $A \cap J_+^M(K)$ is relatively compact.

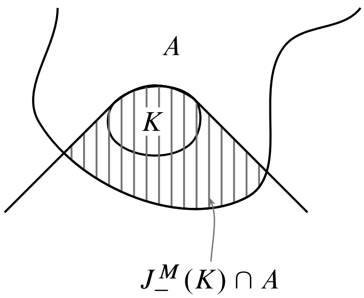


Figure 44. $A \cap J_-^M(K)$ is relatively compact.

Proof. It suffices to consider the first case. The family of open sets $I_-^M(x)$, $x \in M$, is an open covering of M. Since K is compact it is covered by a finite number of such sets,

$$K \subset I_-^M(x_1) \cup \cdots \cup I_-^M(x_l).$$

We conclude

$$J_-^M(K) \subset J_-\big(I_-^M(x_1) \cup \cdots \cup I_-^M(x_l)\big) \subset J_-^M(x_1) \cup \cdots \cup J_-^M(x_l).$$

Since each $A \cap J_-^M(x_j)$ is relatively compact, we have that

$$A \cap J_-^M(K) \subset \bigcup_{j=1}^{l} \left(A \cap J_-^M(x_j)\right)$$

is contained in a compact set. □

Corollary A.5.4. *Let S be a Cauchy hypersurface in a globally hyperbolic Lorentzian manifold M and let $K \subset M$ be compact. Then $J_\pm^M(K) \cap S$ and $J_\pm^M(K) \cap J_\mp^M(S)$ are compact.*

Proof. The causal future of every Cauchy hypersurface is past-compact. This follows e.g. from [O'Neill1983, Chap. 14, Lemma 40]. Applying Lemma A.5.3 to $A := J_+^M(S)$ we conclude that $J_-^M(K) \cap J_+^M(S)$ is relatively compact in M. By [O'Neill1983, Chap. 14, Lemma 22] the subsets $J_\pm^M(S)$ and the causal relation "$\leq$" are closed. By Lemma A.5.1 $J_-^M(K) \cap J_+^M(S)$ is closed, hence compact.

Since S is a closed subset of $J_+^M(S)$ we also have that $J_-^M(K) \cap S$ is compact.

The statements on $J_+^M(K) \cap J_-^M(S)$ and on $J_+^M(K) \cap S$ are analogous. □

Lemma A.5.5. *Let M be a time-oriented convex domain. Then the causal relation "$\leq$" is closed. In particular, the causal future and the causal past of each point are closed subsets of M.*

Proof. Let $p, p_i, q, q_i \in M$ with $\lim_{i\to\infty} p_i = p$, $\lim_{i\to\infty} q_i = q$, and $p_i \leq q_i$ for all i. We have to show $p \leq q$.

Let $x \in T_pM$ be the unique vector such that $q = \exp_p(x)$ and, similarly, for each i let $x_i \in T_{p_i}M$ be such that $\exp_{p_i}(x_i) = q_i$. Since $p_i \leq q_i$ and since $\exp_{p_i}$ maps $J_+(0) \cap \exp_{p_i}^{-1}(M)$ in $T_{p_i}M$ diffeomorphically onto $J_+^M(p_i)$, we have $x_i \in J_+(0)$, hence $\langle x_i, x_i\rangle \leq 0$. From $\lim_{i\to\infty} p_i = p$ and $\lim_{i\to\infty} q_i = q$ we conclude $\lim_{i\to\infty} x_i = x$ and therefore $\langle x, x\rangle \leq 0$. Thus $x \in J_+(0) \cup J_-(0) \subset T_pM$.

Now let T be a smooth vector field on M representing the time-orientation. In other words, T is timelike and future directed. Then $\langle T, x_i\rangle \leq 0$ because x_i is future directed and so $\langle T, x\rangle \leq 0$ as well. Thus $x \in J_+(0) \subset T_pM$ and hence $p \leq q$. □

Lemma A.5.6. *Let M be a time-oriented Lorentzian manifold and let $S \subset M$ be a spacelike hypersurface. Then for every point p in S, there exists a basis of open neighborhoods Ω of p in M such that $S \cap \Omega$ is a Cauchy hypersurface in Ω.*

Proof. Let $p \in S$. Since every spacelike hypersurface is locally acausal there exists an open neighborhood U of p in M such that $S \cap U$ is an acausal spacelike hypersurface of U. Let Ω be the Cauchy development of $S \cap U$ in U.

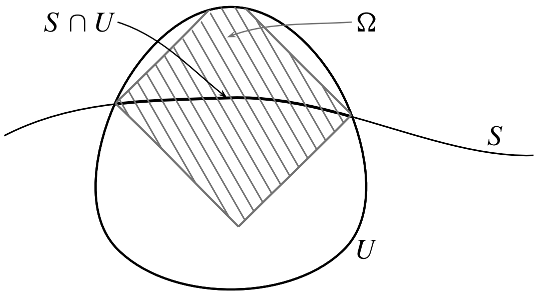


Figure 45. Cauchy development of $S \cap U$ in U.

Since Ω is the Cauchy development of an acausal hypersurface containing p, it is an open neighborhood of p in U and hence also in M. It follows from the definition of the Cauchy development that $\Omega \cap S = S \cap U$ and that $S \cap \Omega$ is a Cauchy hypersurface of Ω.

Given any neighborhood V of p the neighborhood U from above can be chosen to be contained in V. Hence Ω is also contained in V. Therefore we get a basis of neighborhoods Ω with the required properties. □

On globally hyperbolic manifolds the relation $\leq$ is always closed [O'Neill1983, Chap. 14, Lemma 22]. The statement that the sets $J_+^M(p) \cap J_-^M(q)$ are compact can be strengthened as follows:

Lemma A.5.7. *Let $K, K' \subset M$ two compact subsets of a globally hyperbolic Lorentzian manifold M. Then $J_+^M(K) \cap J_-^M(K')$ is compact.*

Proof. Let p in M. By the definition of global-hyperbolicity, the subset $J_+^M(p)$ is past compact in M. It follows from Lemma A.5.3 that $J_+^M(p) \cap J_-^M(K')$ is relatively compact in M. Since the relation $\leq$ is closed on M, the sets $J_+^M(p)$ and $J_-^M(K')$ are closed by Lemma A.5.1. Hence $J_+^M(p) \cap J_-^M(K')$ is actually compact. This holds for every $p \in M$, i.e., $J_-^M(K')$ is future compact in M. It follows again from Lemma A.5.3 that $J_+^M(K) \cap J_-^M(K')$ is relatively compact in M, hence compact by Lemma A.5.1. □

Lemma A.5.8. *Let $\Omega \subset M$ be a nonempty open subset of a time-oriented Lorentzian manifold M. Let $J_+^M(p) \cap J_-^M(q)$ be contained in Ω for all $p, q \in \Omega$. Then Ω is causally compatible.*

If furthermore M is globally hyperbolic, then Ω is globally hyperbolic as well.

Proof. We first show that $J_\pm^M(p) \cap \Omega = J_\pm^\Omega(p)$ for all $p \in \Omega$. Let $p \in \Omega$ be fixed. The inclusion $J_\pm^\Omega(p) \subset J_\pm^M(p) \cap \Omega$ is obvious. To show the opposite inclusion let $q \in J_+^M(p) \cap \Omega$. Then there exists a future directed causal curve $c\colon [0, 1] \to M$ with

$c(0) = p$ and $c(1) = q$. For every $z \in c([0,1])$ we have $z \in J_+^M(p) \cap J_-^M(q) \subset \Omega$, i.e., $c([0,1]) \subset \Omega$. Therefore $q \in J_+^\Omega(p)$. Hence $J_+^M(p) \cap \Omega \subset J_+^\Omega(p)$ and $J_-^M(p) \cap \Omega \subset J_-^\Omega(p)$ can be seen similarly.

We have shown $J_\pm^M(p) \cap \Omega = J_\pm^\Omega(p)$, i.e., Ω is a causally compatible subset of M. Let now M be globally hyperbolic. Then since for any two points $p, q \in \Omega$ the intersection $J_+^M(p) \cap J_-^M(q)$ is contained in Ω the subset

$$J_+^\Omega(p) \cap J_-^\Omega(q) = J_+^M(p) \cap J_-^M(q) \cap \Omega = J_+^M(p) \cap J_-^M(q)$$

is compact. Remark 1.3.9 concludes the proof. □

Lemma A.5.9. *For any acausal hypersurface S in a time-oriented Lorentzian manifold the Cauchy development $D(S)$ is a causally compatible and globally hyperbolic open subset of M.*

Proof. Let S be an acausal hypersurface in a time-oriented Lorentzian manifold M. By [O'Neill1983, Chap. 14, Lemma 43] $D(S)$ is an open and globally hyperbolic subset of M. Let $p, q \in D(S)$. Let $z \in J_+^M(p) \cap J_-^M(q)$. We choose a future directed causal curve c from p through z to q. Extend c to an inextendible causal curve in M, again denoted by c. Since $p \in D(S)$ the curve c must meet S. Since S is acausal this intersection point is unique.

Now let $\tilde{c}$ be any inextendible causal curve through z. If c intersects S before z, then look at the inextendible curve obtained by first following $\tilde{c}$ until z and then following c. This is an inextendible causal curve through q. Since $q \in D(S)$ this curve must intersect S. This intersection point must come before z, hence lie on $\tilde{c}$. Similarly, if c intersects S at or after z, then look at the inextendible curve obtained by first following c until z and then following $\tilde{c}$. Again, this curve is inextendible causal and goes through $p \in D(S)$. Hence it must hit S and this intersection point must come before or at z, thus it must again lie on $\tilde{c}$.

In any case $\tilde{c}$ intersects S. This shows $z \in D(S)$. We have proved $J_+^M(p) \cap J_-^M(q) \subset D(S)$. By Lemma A.5.8 $D(S)$ is causally compatible in M. □

Note furthermore that, by the definition of $D(S)$, the acausal subset S is a Cauchy hypersurface of $D(S)$.

Lemma A.5.10. *Let M be a globally hyperbolic Lorentzian manifold. Let $\Omega \subset M$ be a causally compatible and globally hyperbolic open subset. Assume that there exists a Cauchy hypersurface Σ of Ω which is also a Cauchy hypersurface of M.*

Then every Cauchy hypersurface of Ω is also a Cauchy hypersurface of M.

Proof. Let S be any Cauchy hypersurface of Ω. Since Ω is causally compatible in M, achronality of S in Ω implies achronality of S in M.

Let $c: I \to M$ be any inextendible timelike curve in M. Since Σ is a Cauchy hypersurface of M there exists some $t_0 \in I$ with $c(t_0) \in \Sigma \subset \Omega$. Let $I' \subset I$ be the connected component of $c^{-1}(\Omega)$ containing t_0. Then I' is an open interval and $c|_{I'}$ is an inextendible timelike curve in Ω. Therefore it must hit S. Thus S is a Cauchy hypersurface in M. □

Given any compact subset K of a globally hyperbolic manifold M, one can construct a causally compatible globally hyperbolic open subset of M which is "causally independent" of K:

Lemma A.5.11. *Let K be a compact subset of a globally hyperbolic Lorentzian manifold M. Then the subset $M \setminus J^M(K)$ is, when nonempty, a causally compatible globally hyperbolic open subset of M.*

Proof. Since M is globally hyperbolic and K is compact it follows from Lemma A.5.1 that $J^M(K)$ is closed in M, hence $M \setminus J^M(K)$ is open. Next we show that $J_+^M(x) \cap J_-^M(y) \subset M \setminus J^M(K)$ for any two points $x, y \in M \setminus J(K)$. It will then follow from Lemma A.5.8 that $M \setminus J^M(K)$ is causally compatible and globally hyperbolic.

Let $x, y \in M \setminus J^M(K)$ and pick $z \in J_+^M(x) \cap J_-^M(y)$. If $z \notin M \setminus J^M(K)$, then $z \in J_+^M(K) \cup J_-^M(K)$. If $z \in J_+^M(K)$, then $y \in J_+^M(z) \subset J_+^M(J_+^M(K)) = J_+^M(K)$ in contradiction to $y \notin J^M(K)$. Similarly, if $z \in J_-^M(K)$ we get $x \in J_-^M(K)$, again a contradiction. Therefore $z \in M \setminus J^M(K)$. This shows $J_+^M(x) \cap J_-^M(y) \subset M \setminus J^M(K)$. □

Next we prove the existence of a causally compatible globally hyperbolic neighborhood of any compact subset in any globally hyperbolic manifold. First we need a technical lemma.

Lemma A.5.12. *Let A and B two nonempty subsets of a globally hyperbolic Lorentzian manifold M. Then $\Omega := I_+^M(A) \cap I_-^M(B)$ is a globally hyperbolic causally compatible open subset of M.*

Furthermore, if A and B are relatively compact in M, then so is Ω.

Proof. Since the chronological future and past of any subset of M are open, so is Ω. For any $x, y \in \Omega$ we have $J_+^M(x) \cap J_-^M(y) \subset \Omega$ because $J_+^M(x) \subset J_+^M(I_+^M(A)) = I_+^M(A)$ and $J_-^M(y) \subset J_-^M(I_-^M(B)) = I_-^M(B)$. Lemma A.5.8 implies that Ω is globally hyperbolic and causally compatible.

If furthermore A and B are relatively compact, then

$$\Omega \subset J_+^M(A) \cap J_-^M(B) \subset J_+^M(\bar{A}) \cap J_-^M(\bar{B})$$

and $J_+^M(\bar{A}) \cap J_-^M(\bar{B})$ is compact by Lemma A.5.7. Hence Ω is relatively compact in M. □

Proposition A.5.13. *Let K be a compact subset of a globally hyperbolic Lorentzian manifold M. Then there exists a relatively compact causally compatible globally hyperbolic open subset O of M containing K.*

Proof. Let $h: M \to \mathbb{R}$ be a Cauchy time-function as in Corollary 1.3.12. The level sets $S_t := h^{-1}(\{t\})$ are Cauchy hypersurfaces for each $t \in h(M)$. Since K is compact so is $h(K)$. Hence there exist numbers $t_+, t_- \in h(M)$ such that

$$S_{t_\pm} \cap J_\mp^M(K) = \emptyset,$$

that is, such that K lies in the past of S_{t_+} and in the future of S_{t_-}. We consider the open set

$$O := I_-^M\big(J_+^M(K) \cap S_{t_+}\big) \cap I_+^M\big(J_-^M(K) \cap S_{t_-}\big).$$

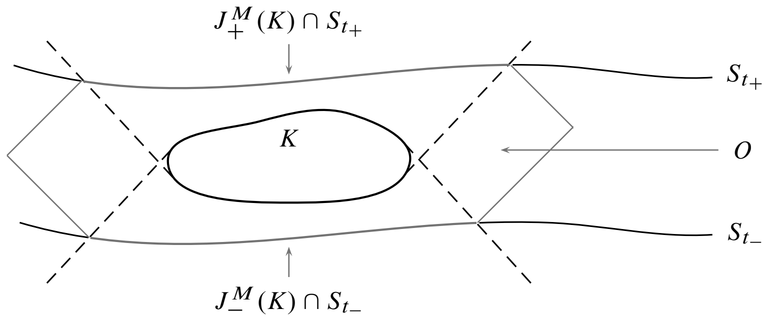


Figure 46. Construction of globally hyperbolic neighborhood O of K.

We show $K \subset O$. Let $p \in K$. By the choice of $t_\pm$ we have $K \subset J_\mp^M(S_{t_\pm})$. Choose any inextendible future directed timelike curve starting at p. Since S_{t_+} is a Cauchy hypersurface, it is hit exactly once by this curve at a point q. Therefore $q \in I_+^M(K) \cap S_{t_+}$ hence $p \in I_-^M(q) \subset I_-^M(I_+^M(K) \cap S_{t_+}) \subset I_-^M(J_+^M(K) \cap S_{t_+})$. Analogously $p \in I_+^M(J_-^M(K) \cap S_{t_-})$. Therefore $p \in O$.

It follows from Lemma A.5.12 that O is a causally compatible globally hyperbolic open subset of M. Since every Cauchy hypersurface is future and past compact, the subsets $J_-^M(K) \cap S_{t_-}$ and $J_+^M(K) \cap S_{t_+}$ are relatively compact by Lemma A.5.3. According to Lemma A.5.12 the subset O is also relatively compact. This finishes the proof. □

Lemma A.5.14. *Let (S, g_0) be a connected Riemannian manifold. Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a smooth positive function. Let $M = I \times S$ and $g = -dt^2 + f(t)^2 g_0$. We give M the time-orientation with respect to which the vector field $\frac{\partial}{\partial t}$ is future directed.*

Then (M, g) is globally hyperbolic if and only if (S, g_0) is complete.

Proof. Let (S, g_0) be complete. Each slice $\{t_0\} \times S$ in M is certainly achronal, $t_0 \in I$. We show that they are Cauchy hypersurfaces by proving that each inextendible causal curve meets all the slices.

Let $c(s) = (t(s), x(s))$ be a causal curve in $M = I \times S$. Without loss of generality we may assume that c is future directed, i.e., $t'(s) > 0$. We can reparametrize c and use t as the curve parameter, i.e., c is of the form $c(t) = (t, x(t))$.

Suppose that c is inextendible. We have to show that c is defined on all of I. Assume that c is defined only on a proper subinterval $(\alpha, \beta) \subset I$ with, say, $\alpha \in I$.

Pick $\varepsilon > 0$ with $[\alpha - \varepsilon, \alpha + \varepsilon] \subset I$. Then there exist constants $C_2 > C_1 > 0$ such that $C_1 \leq f(t) \leq C_2$ for all $t \in [\alpha - \varepsilon, \alpha + \varepsilon]$. The curve c being causal means $0 \geq g(c'(t), c'(t)) = -1 + f(t)^2 \|x'(t)\|^2$ where $\| \cdot \|$ is the norm induced by g_0. Hence $\|x'(t)\| \leq \frac{1}{f(t)} \leq \frac{1}{C_1}$ for all $t \in (\alpha, \alpha + \varepsilon)$.

Now let $(t_i)_i$ be a sequence with $t_i \searrow \alpha$. For sufficiently large i we have $t_i \in (\alpha, \alpha + \varepsilon)$. For $j > i \gg 0$ the length of the part of x from t_i to t_j is bounded from above by $\frac{t_j - t_i}{C_1}$. Thus we have for the Riemannian distance

$$\operatorname{dist}(x(t_j), x(t_i)) \leq \frac{t_j - t_i}{C_1}.$$

Hence $(x(t_i))_i$ is a Cauchy sequence and since (S, g_0) is complete it converges to a point $p \in S$. This limit point p does not depend on the choice of Cauchy sequence because the union of two such Cauchy sequences is again a Cauchy sequence with a unique limit point. This shows that the curve x can be extended continuously by putting $x(\alpha) := p$. We extend x in an arbitrary fashion beyond α to a piecewise C^1-curve with $\|x'(t)\| \leq \frac{1}{C_2}$ for all $t \in (\alpha - \varepsilon, \alpha)$. This yields an extension of c with

$$g(c'(t), c'(t)) = -1 + f(t)^2 \|x'(t)\|^2 \leq -1 + \frac{f(t)^2}{C_2^2} \leq 0.$$

Thus this extension is causal in contradiction to the inextendibility of c.

Conversely, assume that (M, g) is globally hyperbolic. We fix $t_0 \in I$ and choose $\varepsilon > 0$ such that $[t_0 - \varepsilon, t_0 + \varepsilon] \subset I$. There is a constant $\eta > 0$ such that $\frac{1}{f(t)} \geq \eta$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. Fix $p \in S$. For any $q \in S$ with $\operatorname{dist}(p, q) \leq \frac{\varepsilon\eta}{2}$ there is a smooth curve x in S of length at most $\varepsilon\eta$ joining p and q. We may parametrize x on $[t_0, t_0 + \varepsilon]$ such that $x(t_0) = q$, $x(t_0 + \varepsilon) = p$ and $\|x'\| \leq \eta$. Now the curve $c(t) := (t, x(t))$ is causal because

$$g(c', c') = -1 + f^2 \, \|x'\|^2 \leq -1 + f^2 \, \eta^2 \leq 0.$$

Moreover, $c(t_0) = (t_0, q)$ and $c(t_0 + \varepsilon) = (t_0 + \varepsilon, p)$. Thus $(t_0, q) \in J_-^M(t_0 + \varepsilon, p)$. Similarly, one sees $(t_0, q) \in J_+^M(t_0 - \varepsilon, p)$. Hence the closed ball $\bar{B}_r(p)$ in S is contained in the compact set $J_+^M(t_0 - \varepsilon, p) \cap J_-^M(t_0 + \varepsilon, p)$ and therefore compact itself, where $r = \frac{\eta\varepsilon}{2}$. We have shown that all closed balls of the fixed radius $r > 0$ in S are compact.

Every metric space with this property is complete. Namely, let $(p_i)_i$ be a Cauchy sequence. Then there exists $i_0 > 0$ such that $\operatorname{dist}(p_i, p_j) \leq r$ whenever $i, j \geq i_0$. Thus $p_j \in \bar{B}_r(p_{i_0})$ for all $j \geq i_0$. Since any Cauchy sequence in the compact ball $\bar{B}_r(p_{i_0})$ must converge we have shown completeness. □

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Figures

Symbols

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